Spectral Graph Theory		**Winter 2020**
	Problem Set 3	
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Problem 3.1 In class we began to analyze the expected eigenvalues of M, the adjacency matrix of a random graph $G \sim G_n(p)$. We defined the matrix R as follow:

$$R_{i,j} = \begin{cases} 0 & i = j, \\ 1-p & \text{w.p. } p, \\ -p & \text{w.p. } 1-p \end{cases}$$

and gave a bound on the eigenvalues of R that holds with high probability.

Use this to give a lower bound on the maximal eigenvalue of M and an upper bound on the rest of the eigenvalues of M, with the same probability.

Problem 3.2

1. Describe a randomized algorithm that given a PSD matrix M, a positive value $\epsilon \in \mathbb{R}$ and an eigenvector of largest eigenvalue v_1, computes x_k such that $x_k \perp v_1$ and
$$\frac{x_k^T M x_k}{x_k^T x_k} \geq \lambda_2(1 - O(\epsilon)).$$
Your algorithm should succeed with positive probability, and be efficient.
2. Describe a randomized algorithm that given a PSD matrix M, a positive value $\epsilon \in \mathbb{R}$ and an eigenvector of smallest eigenvalue v_n, computes x_k such that $x_k \perp v_1$ and
$$\frac{x_k^T M x_k}{x_k^T x_k} \leq \lambda_{n-1}(1 + O(\epsilon)).$$
Your algorithm should succeed with positive probability, and be efficient.

Problem 3.3 Let A be a nonsingular symmetric real matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. In many cases, one needs to bound the largest eigenvalue of A after a rank one update, i.e., bound the largest eigenvalue of $A + cvv^T$ as a function of the largest eigenvalue of A, the vector v and the scalar c. The largest eigenvalue itself however is difficult to analyze directly. Instead, one looks at "smoother" quantities that interact with the largest eigenvalue. Specifically, define the function

$$\Phi^u(A) = \sum_{i=1}^{n} \frac{1}{u - \lambda_i}$$

(which we will make use of at the end of the course).

1. Prove that $\lambda_n \leq u - 1/\Phi^u(A)$ for all $u > \lambda_n$. (Hence, $\Phi^u(A)$ induces an upper bound on the largest eigenvalue.)
2. Prove the identity
$$\left(A - cvv^T\right)^{-1} = A^{-1} + c \cdot \frac{A^{-1}vv^TA^{-1}}{1 - cv^TA^{-1}v}.$$
3. Prove that
$$\Phi^u(A + cvv^T) = \Phi^u(A) + c \cdot \frac{v^T(uI - A)^{-2}v}{1 - cv^T(uI - A)^{-1}v}.$$

Problem 3.4 Let A be a real symmetric matrix. If A is a PSD we write $A \succeq 0$. We define a partial order on real symmetric matrices as follows. Given A, B, we say that A is larger or equal than B, which we denote by $A \succeq B$, if and only if $A - B \succeq 0$. Given graphs G, H we write $G \succeq H$ if $L_G \succeq L_H$. One last notation, given a graph G and $c > 0$ real, we write $c \cdot G$ for the weighted graph obtained by multiplying every edge of G by c.

1. Prove that if H is a subgraph of G (on the same vertex set with a subset of the edges) then $G \succeq H$.
2. Let G, H be graphs such that $G \succeq c \cdot H$. Prove that $\lambda_k(G) \geq c\lambda_k(H)$ for every k.
3. Prove that if $aH \preceq G \preceq bH$ for graphs G, H on a common vertex set V, then for every $S \subseteq V$, $a\partial_H(S) \leq \partial_G(S) \leq b\partial_H(S)$.
4. Let $G_{\{1,n\}}$ be a graph on n vertices with a single edge $\{1, n\}$. Let P be the path graph from 1 to n. Prove that $(n-1)P \succeq G_{\{1,n\}}$.
5. Let G be an unweighted graph of diameter at most r. Prove that $\lambda_2(G) \geq \frac{2}{r(n-1)}$. *Guidance*: observe that for every edge $\{u, v\}$ in G it holds that

   $$G_{\{u,v\}} \preceq rP_{\{u,v\}} \preceq rG,$$

   where $P_{\{u,v\}}$ is the path graph from u, v of length at most r, and $G_{\{u,v\}}$ is the graph with the single edge $\{u, v\}$.
6. Prove that for every unweighted undirected graph G on n vertices with no isolated vertices, it holds that

   $$\frac{\lambda_i}{d_{\max}} \leq \nu_i \leq \frac{\lambda_i}{d_{\min}},$$

   where $d_{\min}$ and $d_{\max}$ are the smallest and largest degrees in G.
7. Show that a length-t lazy random walk on any connected graph on n vertices converges after $t = n^{O(1)}$ steps. Formally, prove that there exists a constant $c \geq 1$ such that for every graph G as above with a random walk matrix W and every probability distribution p on the n vertices of G, it holds that

   $$\left\| \left(\frac{W+I}{2} \right)^t p - \pi \right\|_1 \leq \frac{1}{n}$$

   where $t = n^c$.