The extreme eigenvalues of the adjacency matrix

Following Spielman, Chapters 4, 19

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November 9, 2020

Overview

First bounds

Let G be an n-vertex undirected graph with adjacency matrix $\mathbf{M}$. We denote $\mathbf{M}$'s eigenvalues by $\mu_1 \geq \cdots \geq \mu_n$. When G is d-regular, $\mu_1 = d$. In general,

Claim

$$d_{avg} \leq \mu_1 \leq d_{max}$$

Tightness

Claim

If G is connected and $\mu_1 = d_{max}$ then G is regular.

An improved bound

Claim

μ_1 is bounded below by the average degree of every subgraph of G.

Overview

Wilf's Theorem

Theorem

Let G be an undirected (simple) graph. Then, its chromatic number $\chi(G) \leq \lfloor \mu_1 \rfloor + 1$.

Background check

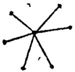
THE EIGENVALUES OF A GRAPH AND ITS CHROMATIC NUMBER

H. S. WILF

Let G be a finite, connected, undirected graph, without loops or multiple edges. If v is a vertex of G, the degree of v, $\rho(v)$, is the number of edges emanating from v. R. L. Brooks has shown [1] that

$$k \leqslant 1 + \max_{v \in G} \rho(v) \tag{1}$$

where k is the chromatic number of G, with equality if and only if G is a complete graph or an odd circuit. The estimator (1) may be crude if G has just a few vertices of high degree. An extreme case is the star graph on n vertices



(2)

for which $k=2$ and (1) gives only $k \leqslant n$. It seems, therefore, desirable to find an upper estimate, of the character of (1), which is more global in nature, and therefore is less sensitive to the idiosyncrasies of a few uninfluential vertices.

Background check

References

1. R. L. Brooks, " On coloring the nodes of a network ", *Proc. Cambridge Phil. Soc.*, **37** (1941), 194–197.
2. G. A. Dirac, " Note on the colouring of graphs ", *Math. Zeitschrift*, 54 (1951), 347–353 .

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Background check

Family [edit]

He was born Balázs Gábor in Budapest, to Richard Balazs and Margit "Manci" Wigner (sister of Eugene Wigner). When his mother married Paul Dirac in 1937, he and his sister resettled in England and were formally adopted, changing their family name to Dirac.

Hoffman's Bound

Let G be a simple undirected graph. The independence number, $\alpha(G)$, is the size of the largest independent set in G. Observe that

$$\alpha(G) \geq \frac{n}{\chi(G)}.$$

Theorem (Hoffman 1970)

Let G be a simple undirected d-regular graph, and μ_n its smallest adjacency matrix eigenvalue. Then,

$$\alpha(G) \leq n \cdot \frac{-\mu_n}{d - \mu_n}.$$

In particular,

$$\chi(G) \geq \frac{d - \mu_n}{-\mu_n}$$

The Godsil-Newman Bound

We will prove the following strengthening.

Theorem (Godsil-Newman 2008)

Let G be a simple undirected graph with largest Laplacian eigenvalue λ_n. Let S an independent set. Then,

$$|S| \leq n\left(1 - \frac{d_{avg}(S)}{\lambda_n}\right),$$

where $d_{avg}(S) = \frac{1}{|S|}\sum_{v \in S} \deg(v)$.

Extra room for the proof

Ramsey Graphs

Definition

An undirected graph on n vertices is k-Ramsey if it contains neither a clique nor an independent set of size k.

Ramsey (1928) proved that $k > \frac{1}{2}\log_2 n$. Erdos (1947) complemented that by proving the existence of k-Ramsey graphs with $k \leq 2\log_2 n$. A major open problem is to come with explicit k-Ramsey graphs for $k = O(\log n)$.

The state-of-the-art constructions are coming from randomness extractors, and achieve $k = \widetilde{O}(\log n)$.

Paley Graphs are OK Ramsey Graphs (and more)

Recall that a Paley graph on $n = p$ vertices is $d = (p-1)/2$ regular with $\lambda_n = (p + \sqrt{p})/2$. A quick calculation shows that if S is an independent set then

$$\begin{aligned} |S| &\leq n\left(1 - \frac{d_{\text{avg}}(S)}{\lambda_n}\right) \\ &= p\left(1 - \frac{p-1}{p+\sqrt{p}}\right) \\ &= \sqrt{p}. \end{aligned}$$

Other Ramsey Graphs

Beautiful, but can you come up with a trivial $\sqrt{n} + 1$ Ramsey graph?

The Paley graph conjecture implies that the actual bound is $p^{o(1)}$ rather than $\sqrt{p}$.

Other Ramsey Graphs

One more neat construction due to Bourgain (2005) for $n = 2^k$. Identify the vertices with both $\mathbb{F}_2^k$ and $\mathbb{F}_{2^k}$. Connect x, y if and only if

$$\langle x^3 + x, y^3 + y \rangle = 1.$$

This gives $n^{\frac{1}{2}-\varepsilon}$ for some tiny constant $\varepsilon > 0$.

If you are interested hearing more about Ramsey graph constructions see (among other) my talk at IAS (links are available from my homepage).

The Perron-Frobenius Theorem (for symmetric matrices)

Theorem

Let G be a connected graph with adjacency matrix $\mathbf{M}$ and corresponding eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then,

- μ_1 *has a strictly positive eigenvector.*
- $\mu_1 \geq -\mu_n$. *Equality holds if and only if G is bipartite.*
- $\mu_1 > \mu_2$.

Extra room for the proof

Extra room for the proof