Recitation 4: Circuits

1. Calculate and describe what the circuit in Figure 4.1 does. (Question 1 in the Circuits part of Ex3)

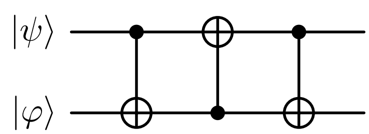

Figure 4.1: Circuit on 2 qubits. the gates are all CNOT, where the • is the control qubit and the ⊕ is the target qubit.

   Answer. Note that CNOT can be written for computational basis elements as $CNOT_{1,2}\left|xy\right\rangle_{1,2} = \left|x(x \oplus y)\right\rangle$. Then the circuits act on the computational basis as $C\left|xy\right\rangle = CNOT_{1,2}CNOT_{2,1}CNOT_{1,2}\left|xy\right\rangle = CNOT_{1,2}CNOT_{2,1}\left|x(x\oplus y)\right\rangle = CNOT_{1,2}\left|(x\oplus x\oplus y)(x\oplus y)\right\rangle = CNOT_{1,2}\left|y(x\oplus y)\right\rangle = \left|y(y\oplus x\oplus y)\right\rangle = \left|yx\right\rangle$. By linearity, for general $\left|\psi\right\rangle = \alpha_0\left|0\right\rangle + \alpha_1\left|1\right\rangle$ and $\left|\varphi\right\rangle = \beta_0\left|0\right\rangle + \beta_1\left|1\right\rangle$ we get $C\left|\psi\right\rangle\left|\varphi\right\rangle = C(\alpha_0\beta_0\left|00\right\rangle + \alpha_0\beta_1\left|01\right\rangle + \alpha_1\beta_0\left|10\right\rangle + \alpha_1\beta_1\left|11\right\rangle) = \alpha_0\beta_0\left|00\right\rangle + \alpha_0\beta_1\left|10\right\rangle + \alpha_1\beta_0\left|01\right\rangle + \alpha_1\beta_1\left|11\right\rangle = \left|\varphi\right\rangle\left|\psi\right\rangle$. So this operator swaps between the qubit states and is called SWAP. △

2. We said we can measure in any orthonormal basis, and even in any orthogonal decomposition of the space. But the circuit formalism normally uses measurements in the computational basis. How can we implement a measurement in another basis $\{\left|v_i\right\rangle\}_{i=0}^{d-1}$ using unitary transformations and computational-basis measurements? (question 2 in the Circuits part of Ex3)

   Answer. A measurement in some orthonormal basis $\{\left|v_i\right\rangle\}_{i=0}^{d-1}$ is equivalent to performing the unitary $U : \left|v_i\right\rangle \mapsto \left|i\right\rangle$ that maps this basis to the computational basis, measuring in the computational basis and finally applying the inverse $U^\dagger$:

   Let $\left|\psi\right\rangle = \sum_{i=0}^{d-1}\alpha_i\left|v_i\right\rangle$ be any quantum state. $U\left|\psi\right\rangle = \sum_{i-=0}^{d-1}\alpha_i\left|i\right\rangle$ and measuring it in the computational basis yields the "classical outcome"

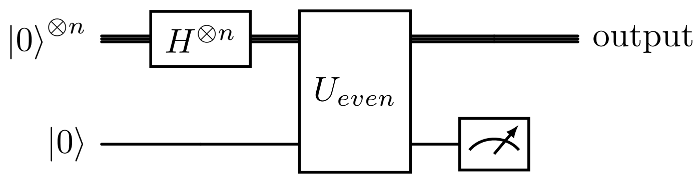

Figure 4.2: Circuit that probabilistically prepares the uniform superposition over the even numbers (when the outcome is $|1\rangle$).

i and the "quantum outcome" $\frac{\alpha_i}{|\alpha_i|}|i\rangle$ with probability $|\alpha_i|^2$. So we get the "classical outcome" in the correct probability, but we are left with the wrong quantum state. Applying $U^\dagger$ fixes that as well as we get $U^\dagger \frac{\alpha_i}{|\alpha_i|}|i\rangle = \frac{\alpha_i}{|\alpha_i|}|v_i\rangle$. △

Let's generalise this question to the slightly more general measurement, by an example: create a circuit that prepares, with success probability $\frac{1}{2}$, the state we looked at in the last recitation: $\frac{\sqrt{2}}{\sqrt{d}}\sum_{\substack{i\in\{0,\dots,d-1\} \\ i\equiv 0 \mod 2}}|i\rangle$, where $d := 2^n$, for n qubits, and $|i\rangle$ is the computational basis that is i in binary. Assume you can have a unitary U_f on $n+1$ qubits, for any boolean function $f : \{0,1\}^n \to \{0,1\}$ that you know how to calculate classically, such that $U_f|xy\rangle = |x(f(x)\oplus y)\rangle$ for all $x \in \{0,1\}^n, y \in \{0,1\}$.

Answer. Define U_{even} for $even(x) = x + 1 \mod 2$ where x on the RHS here is the integer that x represents in binary. The circuit is draw in Figure 4.2. △