

Adeles

Unit 13

Gil Cohen

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Let K be a field, and R a ring (commutative with unit, as usual). If R is also a K -vector space we say that R is a **K -algebra**.

The definition of an algebra is in fact more general but the above works for us.

A typical example is the ring of polynomials $K[x_1, \dots, x_n]$.

Definition 1 (Adeles)

An **adele** of a function field F/K is a map $\alpha : \mathbb{P} \rightarrow F$ satisfying $v_{\mathfrak{p}}(\alpha(\mathfrak{p})) \geq 0$ almost always.

Remarks.

- We will write $\alpha_{\mathfrak{p}}$ for $\alpha(\mathfrak{p})$.
- The set $\mathbb{A} = \mathbb{A}_{F/K}$ of adeles of F/K is a ring with point-wise addition and multiplication.

$$\begin{aligned}(\alpha\beta)(\mathfrak{p}) &= \alpha(\mathfrak{p})\beta(\mathfrak{p}), \\(\alpha + \beta)(\mathfrak{p}) &= \alpha(\mathfrak{p}) + \beta(\mathfrak{p}).\end{aligned}$$

Definition 2 (Adeles)

An **adele** of a function field F/K is a map $\alpha : \mathbb{P} \rightarrow F$ satisfying $v_p(\alpha(p)) \geq 0$ almost always.

Remarks.

- We identify F as a subfield of \mathbb{A} via

$$x \mapsto [x] \text{ s.t. } [x]_p = x \quad \forall p \in \mathbb{P}.$$

- $\mathbb{A}_{F/K}$ is an F -algebra: For $x \in F$

$$x\alpha = [x]\alpha.$$

More explicitly,

$$(x\alpha)_p = x \cdot \alpha_p.$$

- We extend the valuation v_p from F to \mathbb{A} by

$$v_p(\alpha) = v_p(\alpha_p).$$

Recall that for $\mathfrak{a} \in \mathcal{D}$ we defined the Riemann-Roch space

$$\mathcal{L}(\mathfrak{a}) = \{x \in F \mid v_p(x) + v_p(\mathfrak{a}) \geq 0 \text{ for all } p \in \mathbb{P}\}.$$

Definition 3

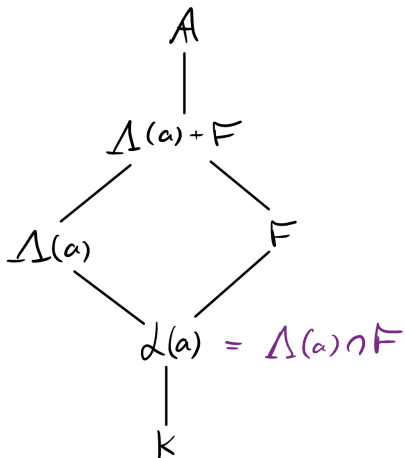
For a divisor \mathfrak{a} let

$$\Lambda(\mathfrak{a}) = \{\alpha \in \mathbb{A} \mid v_p(\alpha) + v_p(\mathfrak{a}) \geq 0 \text{ for all } p \in \mathbb{P}\}.$$

Proving the following claim is left as an exercise.

Claim 4

- $\Lambda(\mathfrak{a})$ is a K -vector space, a subspace of \mathbb{A} .
- $\mathcal{L}(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap F$.



Claim 5

For divisors \mathfrak{a} , \mathfrak{b} and $x \in F$

- $\mathfrak{a} \leq \mathfrak{b} \implies \Lambda(\mathfrak{a}) \subseteq \Lambda(\mathfrak{b})$.
- $\Lambda(\mathfrak{a}) \cap \Lambda(\mathfrak{b}) = \Lambda(\min(\mathfrak{a}, \mathfrak{b}))$.
- $\Lambda(\mathfrak{a}) + \Lambda(\mathfrak{b}) = \Lambda(\max(\mathfrak{a}, \mathfrak{b}))$.
- $x\Lambda(\mathfrak{a}) = \Lambda(\mathfrak{a} - (x))$.

The proof is left as an exercise.

Recall that for divisors $\mathfrak{a} \leq \mathfrak{b}$,

$$\dim_{\mathbb{K}} \left(\mathcal{L}(\mathfrak{b}) / \mathcal{L}(\mathfrak{a}) \right) \leq \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Moreover, if $S \subset \mathbb{P}$ is the prime divisors appearing in at least one of \mathfrak{a} , \mathfrak{b} , then

$$\dim_{\mathbb{K}} \left(\mathcal{L}(\mathfrak{b}, S) / \mathcal{L}(\mathfrak{a}, S) \right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Lemma 6

For divisors $\mathfrak{a} \leq \mathfrak{b}$,

$$\dim_{\mathbb{K}} \left(\Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) \right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Proof.

Let $S \subset \mathbb{P}$ be the prime divisors appearing in at least one of $\mathfrak{a}, \mathfrak{b}$. We proved that

$$\dim_K \left(\mathcal{L}(\mathfrak{b}, S) / \mathcal{L}(\mathfrak{a}, S) \right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

So it suffices to establish the isomorphism (as K -vector spaces)

$$\Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) \cong \mathcal{L}(\mathfrak{b}, S) / \mathcal{L}(\mathfrak{a}, S).$$

Let $T : F \rightarrow \mathbb{A}$ be the map $x \mapsto T(x)$ that is given by

$$T(x)_{\mathfrak{p}} = \begin{cases} x & \text{if } \mathfrak{p} \in S; \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$T(x)_p = \begin{cases} x & \text{if } p \in S; \\ 0 & \text{otherwise.} \end{cases}$$

T is a K -linear map that maps $\mathcal{L}(\mathfrak{a}, S)$ into $\Lambda(\mathfrak{a})$ and $\mathcal{L}(\mathfrak{b}, S)$ into $\Lambda(\mathfrak{b})$.
Indeed,

$$\begin{aligned} x \in \mathcal{L}(\mathfrak{a}, S) &\implies \forall p \in S \quad v_p(x) + v_p(\mathfrak{a}) \geq 0 \\ &\implies \forall p \in \mathbb{P} \quad v_p(T(x)) + v_p(\mathfrak{a}) \geq 0 \\ &\implies T(x) \in \Lambda(\mathfrak{a}). \end{aligned}$$

T composed with the projection map $\Lambda(\mathfrak{b}) \rightarrow \Lambda(\mathfrak{b})/\Lambda(\mathfrak{a})$ gives rise to a map \bar{T} .

$$F \xrightarrow{T} A$$

$$\begin{array}{ccc}
 \mathcal{L}(b, s) & \xrightarrow{T} & \Lambda(b) \\
 \downarrow & \searrow \bar{T} & \downarrow \\
 \mathcal{L}(b, s) / \mathcal{L}(a, s) & \longrightarrow & \Lambda(b) / \Lambda(a)
 \end{array}$$

Proof.

We first prove that $\ker \bar{T} = \mathcal{L}(\mathfrak{a}, S)$. Indeed,

$$\begin{aligned} \ker \bar{T} &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid T(x) \in \Lambda(\mathfrak{a})\} \\ &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid \forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(T(x)) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\}. \end{aligned}$$

For $\mathfrak{p} \in \mathbb{P} \setminus S$, $v_{\mathfrak{p}}(T(x)) = v_{\mathfrak{p}}(0) = \infty$ and $v_{\mathfrak{p}}(\mathfrak{a}) = 0$, and so

$$\begin{aligned} \ker \bar{T} &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid \forall \mathfrak{p} \in S \quad v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\} \\ &= \mathcal{L}(\mathfrak{a}, S). \end{aligned}$$

Proof.

Next, we show that \bar{T} is onto.

Take $\beta \in \Lambda(\mathfrak{b})$. By WAT $\exists x \in F$ s.t.

$$\forall \mathfrak{p} \in S \quad v_{\mathfrak{p}}(T(x) - \beta_{\mathfrak{p}}) = v_{\mathfrak{p}}(x - \beta_{\mathfrak{p}}) \geq -v_{\mathfrak{p}}(\mathfrak{a}),$$

As for $\mathfrak{p} \in \mathbb{P} \setminus S$,

$$v_{\mathfrak{p}}(T(x) - \beta_{\mathfrak{p}}) = v_{\mathfrak{p}}(\beta_{\mathfrak{p}}) \geq -v_{\mathfrak{p}}(\mathfrak{b}) = 0 = -v_{\mathfrak{p}}(\mathfrak{a}).$$

So, $T(x) - \beta \in \Lambda(\mathfrak{a})$. Hence, $\bar{T}(x) = \beta + \Lambda(\mathfrak{a})$.

We just need to now that $x \in \mathcal{L}(\mathfrak{b}, S)$. Indeed, for $\mathfrak{p} \in S$,

$$v_{\mathfrak{p}}(x) \geq \min(v_{\mathfrak{p}}(x - \beta_{\mathfrak{p}}), v_{\mathfrak{p}}(\beta_{\mathfrak{p}})) \geq \min(-v_{\mathfrak{p}}(\mathfrak{a}), -v_{\mathfrak{p}}(\mathfrak{b})) = -v_{\mathfrak{p}}(\mathfrak{b}).$$

The proof then follows by the first isomorphism theorem for vector spaces. □

For divisors $\mathfrak{a} \leq \mathfrak{b}$ we just proved that

$$\dim_{\mathbb{K}} \left(\Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) \right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Lemma 7

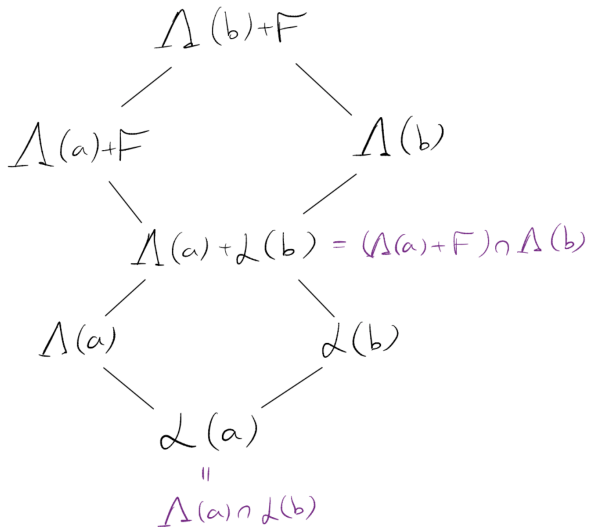
For divisors $\mathfrak{a} \leq \mathfrak{b}$,

$$\dim_{\mathbb{K}} \left(\Lambda(\mathfrak{b}) + F / \Lambda(\mathfrak{a}) + F \right) = (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}).$$

For the proof of Lemma 7 we recall the third isomorphism theorem for vector spaces.

Let $U \subseteq V \subseteq W$ be \mathbb{K} -vector spaces. Then, V/U is a subspace of W/U , and

$$(W/U) / (V/U) \cong W/V.$$



Proof.

By Claim 4,

$$\mathcal{L}(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap \mathfrak{F}.$$

But since $\mathcal{L}(\mathfrak{a}) \subseteq \mathcal{L}(\mathfrak{b})$,

$$\begin{aligned}\mathcal{L}(\mathfrak{a}) &= \mathcal{L}(\mathfrak{a}) \cap \mathcal{L}(\mathfrak{b}) \\ &= \Lambda(\mathfrak{a}) \cap \mathfrak{F} \cap \mathcal{L}(\mathfrak{b}) \\ &= \Lambda(\mathfrak{a}) \cap \mathcal{L}(\mathfrak{b}).\end{aligned}$$

Proof.

We turn to prove that

$$(\Lambda(\mathfrak{a}) + F) \cap \Lambda(\mathfrak{b}) = \Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b}).$$

The \supseteq direction is easy.

Now, take an element in the LHS and write it as $\alpha + x$ with $\alpha \in \Lambda(\mathfrak{a})$ and $x \in F$. But $\Lambda(\mathfrak{a}) \subseteq \Lambda(\mathfrak{b})$ and so

$$\alpha + x \in \Lambda(\mathfrak{b}) \implies x \in \Lambda(\mathfrak{b}).$$

But $x \in F$ and so

$$x \in \Lambda(\mathfrak{b}) \cap F = \mathcal{L}(\mathfrak{b}).$$

Thus,

$$\alpha + x \in \Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b}).$$

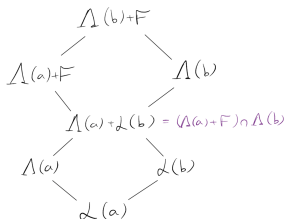
Proof.

By the diagram and by the second isomorphism theorem,

$$(\Lambda(\mathfrak{b}) + \mathfrak{F}) / (\Lambda(\mathfrak{a}) + \mathfrak{F}) \cong \Lambda(\mathfrak{b}) / (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})),$$

and by the third and second isomorphism theorems,

$$\begin{aligned} \Lambda(\mathfrak{b}) / (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})) &\cong \Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) / (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})) / \Lambda(\mathfrak{a}) \\ &\cong \Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) / \mathcal{L}(\mathfrak{b}) / \mathcal{L}(\mathfrak{a}) \end{aligned}$$



Proof.

So,

$$\begin{aligned}\dim_{\mathbb{K}} \left(\Lambda(\mathfrak{b}) + \mathbb{F} / \Lambda(\mathfrak{a}) + \mathbb{F} \right) &= \dim_{\mathbb{K}} \Lambda(\mathfrak{b}) / \Lambda(\mathfrak{a}) - \dim_{\mathbb{K}} \mathcal{L}(\mathfrak{b}) / \mathcal{L}(\mathfrak{a}) \\ &= \deg \mathfrak{b} - \deg \mathfrak{a} - (\dim \mathfrak{b} - \dim \mathfrak{a}) \\ &= (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}).\end{aligned}$$



Corollary 8

$$\deg \mathfrak{a} - \dim \mathfrak{a} = g - 1 \implies \Lambda(\mathfrak{a}) + F = \mathbb{A}.$$

Proof.

Take any divisor $\mathfrak{b} \geq \mathfrak{a}$. By monotonicity,

$$g - 1 \geq \deg \mathfrak{b} - \dim \mathfrak{b} \geq \deg \mathfrak{a} - \dim \mathfrak{a} = g - 1.$$

Lemma 7 then implies that

$$\Lambda(\mathfrak{b}) + F = \Lambda(\mathfrak{a}) + F.$$

Now, take $\alpha \in \mathbb{A}$ and choose $\mathfrak{b} \geq \mathfrak{a}$ s.t.

$$\forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(\mathfrak{b}) + v_{\mathfrak{p}}(\alpha) \geq 0.$$

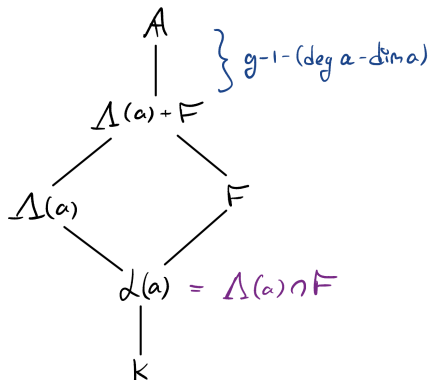
Thus,

$$\alpha \in \Lambda(\mathfrak{b}) \subseteq \Lambda(\mathfrak{a}) + F.$$

Corollary 9

For every divisor α ,

$$\dim_K \mathbb{A} / (\Lambda(\alpha) + F) = g - 1 - (\deg \alpha - \dim \alpha).$$



Proof.

Take any divisor $\mathfrak{b} \geq \mathfrak{a}$ with

$$\deg \mathfrak{b} - \dim \mathfrak{b} = g - 1.$$

By Corollary 8,

$$\Lambda(\mathfrak{b}) + F = \mathbb{A}.$$

Lemma 7 then yields

$$\begin{aligned} \dim_{\mathbb{K}} \mathbb{A} / (\Lambda(\mathfrak{a}) + F) &= \dim_{\mathbb{K}} \left(\Lambda(\mathfrak{b}) + F / \Lambda(\mathfrak{a}) + F \right) \\ &= (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}) \\ &= g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}). \end{aligned}$$