Function Fields

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June 10, 2019

Recall

Definition

Let L/K be be a field extension. $\mathcal{V}(L/K)$ is the set of surjective valuations over L that are trivial on K.

Definition

Let L/K be a field extension of transcendence degree 1. The pair $(\mathcal{V}(L/K), L/K)$ is called a nonsingular complete curve.

We think of $\mathcal{V}(L/K)$ as the "points of the curve X" and of L as (partially defined) functions on X.

It is typical to denote L by $K(X)$. With this notation, the nonsingular complete curve is denoted by $(X, K(X)/K)$ or X/K (even X) in short.

Definition

Let X/K be a nonsingular complete curve. By definition, to a point $P \in X$ corresponds a valuation $v_P \in \mathcal{V}(K(X)/K)$. We proved that to v_P corresponds a local PID $\mathcal{O}_{v_P} \subset K(X)$ which we denote by $\mathcal{O}_P$. The maximal ideal of $\mathcal{O}_P$ is denoted by $\mathcal{M}_P$.

- The ring $\mathcal{O}_P$ is called the ring of rational functions defined at P.
- An element in $\mathcal{O}_P$ is called a function on X defined at P.
- A function $\alpha \in \mathcal{O}_P$ is said to have a zero at P if $\alpha \in \mathcal{M}_P$.

Definition

- For $\alpha \in \mathcal{M}_P$, the integer $v_P(\alpha)$ is called the order of vanishing of α at P.
- A function $\alpha \in K(X) \setminus \mathcal{O}_P$ is said to have a pole at P.
- For such α, the integer $-v_P(\alpha)$ is called the order of the pole of α at P.

Definition

The domain of $\alpha \in K(X)$, denoted by $\mathrm{Dom}(\alpha)$ is the set of points in X where α is defined. That is

$$\mathrm{Dom}(\alpha) = \{P \in X \mid v_P(\alpha) \geq 0\}.$$

For $U \subseteq X$ we let

$$\mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_P$$

be the ring of functions on X defined everywhere on U.

Definition

Let X/K be a nonsingular complete curve. Let $P \in X$ and v the corresponding valuation. We define the residue field

$$K_v = \mathcal{O}_v / \mathcal{M}_v.$$

Claim

$K \hookrightarrow K_v$.

Proof.

The inclusion is via the natural map $k \mapsto k + M_v$. Fix $k \in K^\times$. We have that $v(k) > 0$ and so $k \in \mathcal{O}_v \setminus \mathcal{M}_v$. Therefore, the map is well-defined and injective. □

Definition

An element $\alpha \in L$ is thought of as a partially-defined function

$$\alpha : X \to K_v \cup \{\perp\}$$

that is defined as follows. Let $P \in X$ and $v \in \mathcal{V}(L/K)$ the corresponding valuation. We define

$$\alpha(P) = \begin{cases} \alpha + \mathcal{M}_v, & v(\alpha) \geq 0; \\ \perp, & v(\alpha) < 0. \end{cases}$$

Claim (Straightforward)

For $\alpha, \beta \in \mathcal{O}_v$ it holds that

$$(\alpha + \beta)(P) = \alpha(P) + \beta(P)$$
$$(\alpha\beta)(P) = \alpha(P)\beta(P)$$

Definition

Let L/K be a field extension. The field K is algebraically closed in L if every element of L that is algebraic over K is contained in K. Put differently, if $\bar{K}$ is a fixed choice for an algebraic closure of K and $L \subseteq \bar{K}$ then $\bar{K} \cap L = K$.

Theorem

Let K be a perfect field. Let $f(x, y) \in K[x, y]$ irreducible. Then,

$$K = \bar{K} \cap L \quad \Longleftrightarrow \quad f \text{ is absolutely irreducible.}$$

Proposition

Let L/K be a field extension of transcendence degree 1. Let $K' = \bar{K} \cap L$. Then,

$$K' = \mathcal{O}_X(X).$$

Moreover, $[K' : K] < \infty$.

Corollary

If K is algebraically closed then $\mathcal{O}_X(X) = K$.

Idea.

We omit the proof (next course!) though we have all the tools to prove it. Still some ideas that go into the proof:

- The fact that $[K' : K] < \infty$ is an easy corollary from the following standard claim: if E/K is algebraic then $[E(x) : K(x)] = [E : K]$ (including the "infinity case").
- The $\subseteq$ direction in the asserted equality is easy. The other direction is nontrivial. The idea is to take a supposedly existing $\alpha \in \mathcal{O}_X(X) \setminus K'$, consider the factorization of $\langle \frac{1}{\alpha} \rangle$ in the integral closure of $K[\frac{1}{\alpha}]$ in L, and extract from it a valuation such that $v(\alpha) < 0$. The hard part is to prove such a factorization exists.

□

Definition

Let L/K be a field extension. L is called a function field over K if

1. L/K has transcendence degree 1, and
2. K is algebraically closed in L (that is, $\bar{K} \cap L = K$).

Definition

Let K be any field. A nonsingular complete curve X/K over K is **redefined** to be a nonsingular complete curve as before but with the additional requirement that $\mathcal{O}_X(X) = K$.

Remark

Note then that function fields and nonsingular complete curves is the same thing.