Exercise 1: BraKet, Linear Algebra

1.1 Getting used to BraKet (Dirac) notation

1. Rewrite the following expressions, equations and statements using as much bra-ket notation as you can (try using understanding of the expressions to give short bra-ket description for them):

   (a) $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   (d) $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

   (e) $\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

   (f) The identity matrix Id_n

   (g) $\forall v \in V, \langle v, v \rangle \geq 0$ (meaning real and non-negative)

   (h) Every unitary matrix has an orthonormal basis of eigenvectors, with eigenvalues on the unit circle ($e^{i\theta}$ for some $\theta \in \mathbb{R}$).

   (i) For any two matrices, $A \in M_{n\times m}(\mathbb{C}), B \in M_{m\times n}(\mathbb{C})$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

2. What are the relations between any two of the following:

   (a) For any $v, w \in V$, the relation between $\langle v|w\rangle$ and $\langle w|v\rangle$

(b) For any $v \in V$, the relation between $|v\rangle$ and $\langle v|$

(c) For any two orthonormal bases $v_1 \ldots, v_n$ and $w_1, \ldots, w_n$ of V, the relation between $\sum_{i=1}^{n} |v_i\rangle\langle w_i|$ and $\sum_{i=1}^{n} |w_i\rangle\langle v_i|$

(d) For any $v, w \in V$ and matrices A, B, C with appropriate dimensions, the relation between $(((\langle v| A)B)C) |w\rangle$, $\langle v| (A(B(C |w\rangle)))$ and $\langle v| (((AB)C) |w\rangle)$

3. Prove the statements in subquestions item 1e, item 1g,item 1i using the Bra-Ket notation as much as possible

1.2 Recalling Linear Algebra

Use Bra-Ket notation as much as you can to prove the statements in this section.

1. Prove that the following are equivalent, for a linear operator $U \in L(\mathbb{C}^d)$ over a finite dimensional complex Euclidean space:

   (a) $U^\dagger = U^{-1}$

   (b) U preserves norms: for every vector $|\psi\rangle \in \mathbb{C}^d$, $\||\psi\rangle\| = \|U |\psi\rangle\|$

   (c) U preserves inner products: for every pair of vectors $|\psi\rangle, |\varphi\rangle \in \mathbb{C}^d$, $\langle\psi|\varphi\rangle = \langle\psi| U^\dagger \cdot U |\varphi\rangle$

   (d) U maps any orthonormal basis to another orthonormal basis

   (e) There exists an orthonormal basis which U maps to another orthonormal basis

   (f) The columns of U are an orthonormal basis for $\mathbb{C}^d$

   (g) The rows of U are an orthonormal basis for $\mathbb{C}^d$

   (h) U has an orthonormal eigenbasis, i.e. an orthonormal basis of $\mathbb{C}^d$ which consists of eigenvectors of U, with eigenvalues from the unit circle $e^{i\theta}, \theta \in \mathbb{R}$
   You may use the spectral theorem without proving it, which states that a normal operator $TT^\dagger = T^\dagger T$ has an orthonormal eigenbasis

   Recall that such an operator is called **Unitary**. These are the allowed operations on quantum states, together with measurements.

2. Prove that the following are equivalent, for a linear operator $H \in L(\mathbb{C}^d)$ over a finite dimensional complex Euclidean space:

   (a) $H^\dagger = H$

   (b) H has an orthonormal eigenbasis, i.e. an orthonormal basis of $\mathbb{C}^d$ which consists of eigenvectors of H, with real eigenvalues
   Do NOT use the spectral theorem here. This is a simpler case that can be used to prove the spectral theorem.

Recall that such an operator is called **Hermitian**. These model measurements in a way we'll see later on. (They are also the way we represent more general states, called "mixed states", but we'll get to that much later on.)

3. Prove that two eigenvectors $|\psi\rangle, |\varphi\rangle$ of an Hermitian H with different eigenvalues $\lambda \neq \eta$ must be orthogonal $\langle\psi|\varphi\rangle = 0$.