Unique Factorization of Ideals

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Definition

Let I, J be ideals in a ring A.

- The sum $I + J$ is the ideal $\{i + j \mid i \in I, j \in J\}$.
- The product IJ is the ideal generated by $\{ij \mid i \in I, j \in J\}$.
- For an integer $n \geq 0$ we define J^n recursively by $J^n = J^{n-1}J$, and $J^0 = A$.
- I, J are coprime if $I + J = A$.

Claim

For every ideals I, J in A it holds that

- $IJ \subseteq I \cap J$
- I, J *coprime* $\implies IJ = I \cap J$
- *Distinct maximal ideals are coprime.*
- *If* $IJ \subseteq P \in \text{Spec}(A)$ *then* $I \subseteq P$ *or* $J \subseteq P$.

Definition

A domain A is have the property of unique factorization of ideals (UFI) if

- every ideal $I \neq A$ can be written as $I = P_1 \cdots P_s$, with $P_i \in \mathrm{Spec}(A)$ (not necessarily distinct), and
- this factorization is unique up to a permutation.

Example

Every PID is a UFI.

Claim

Let I be a nontrivial ideal in a noetherian ring A. Then, $\exists P_1, \ldots, P_s \in \mathrm{Spec}(A)$ and integers $a_1, \ldots, a_s \geq 1$ s.t.

$$P_1^{a_1} \cdots P_s^{a_s} \subseteq I \subseteq P_1 \cap \cdots \cap P_s.$$

Proof

Let Σ be the set of ideals in A that do not have the above property. Assume $\Sigma \neq \emptyset$. A noetherian $\implies$ there exists a maximal element $I \in \Sigma$. Clearly, $I \notin \mathrm{Spec}(A)$. Thus, $\exists x, y \in A \setminus I$ s.t. $xy \in I$. Write

$$I_x = \langle I, x \rangle,$$
$$I_y = \langle I, y \rangle.$$

Then,

$$I_x I_y = \langle I^2, Ix, Iy, xy \rangle \subseteq I \subseteq I_x \cap I_y.$$

Proof.

$$I_x I_y = \langle I^2, Ix, Iy, xy\rangle \subseteq I \subseteq I_x \cap I_y.$$

In particular, I_x, I_y are nontrivial ideals. Thus, $I_x, I_y \notin \Sigma$. Therefore,

$$\begin{aligned} P_1^{a_1}\cdots P_s^{a_s} &\subseteq I_x \subseteq P_1 \cap \cdots \cap P_s, \\ Q_1^{b_1}\cdots Q_t^{b_t} &\subseteq I_y \subseteq Q_1 \cap \cdots \cap Q_t. \end{aligned}$$

Thus,

$$\begin{aligned} & I \subseteq I_x \cap I_y \subseteq P_1 \cap \cdots \cap P_s \cap Q_1 \cap \cdots \cap Q_t, \\ & P_1^{a_1}\cdots P_s^{a_s} Q_1^{b_1}\cdots Q_t^{b_t} \subseteq I_x I_y \subseteq I, \end{aligned}$$

contradicting $I \in \Sigma$.

$\square$

Claim

Let A be a noetherian domain of dimension 1. Let $I \neq \langle 0 \rangle, A$ ideal. Then, there are finitely many $M_1, \ldots, M_s \in \mathrm{Max}(A)$ that contains I. Furthermore, $\exists a_1, \ldots, a_s \geq 1$ integers s.t.

$$M_1^{a_1} \cdots M_s^{a_s} \subseteq I \subseteq M_1 \cdots M_s.$$

Proof.

By the previous claim and since $\dim(A) = 1$, $\exists M_1, \ldots, M_s \in \mathrm{Max}(A)$ s.t.

$$M_1^{a_1} \cdots M_s^{a_s} \subseteq I \subseteq M_1 \cap \cdots \cap M_s.$$

Since maximal ideals are coprime, $M_1 \cap \cdots \cap M_s = M_1 \cdots M_s$. Finally, if $I \subseteq M \in \mathrm{Max}(A)$ then $M_1^{a_1} \cdots M_s^{a_s} \subseteq M$. Since M is prime, $M_i \subseteq M$ for some $i \in [s]$, and so, $M = M_i$. □

Claim

Let A be a noetherian domain of dimension 1. TFAE:

- *A is a UFI*
- *A_M is a UFI for all $M \in \mathrm{Max}(A)$.*

Before proving the claim, we state and prove a couple of claims.

Claim

Let A be a domain and $S \subset A$ multiplicative. Let I, J ideals in A. Then,

$$S^{-1}(IJ) = (S^{-1}I)(S^{-1}J).$$

Claim

Let A be a domain. Let $M \in \mathrm{Max}(A)$ and I ideal in A that is not contained in M. Then, $I_M = A_M$.

Claim

Let A be a domain and $S \subset A$ multiplicative. Then, for every ideal $I \in S^{-1}A$,

$$S^{-1}\left(j_S^{-1}(I)\right) = I.$$

Proof

Note that

$$j^{-1}(I) = \left\{a \in A \mid \frac{a}{1} \in I\right\}.$$

Take $i \in I$. Then, $i = \frac{a}{s}$ for $a \in A, s \in S$. Hence,

$$\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in I.$$

Thus,

$$a \in j^{-1}(I) \implies i = \frac{a}{s} \in S^{-1}(j^{-1}(I)).$$

Proof.

This shows that $I \subseteq S^{-1}(j^{-1}(I))$. Now, an element in $S^{-1}(j^{-1}(I))$ is of the form $\frac{a}{s}$ where $\frac{a}{1} \in I$ and $s \in S$. Thus,

$$\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in I.$$

□

Remark

Let I ideal in A, and $S \subset A$ multiplicative. Then,

$$I \subseteq j^{-1}(S^{-1}I)$$

but equality does not necessarily holds.

Claim

Let $J \subseteq I$ be two ideals in a ring A. Then, $J = I$ if and only if $J_M = I_M$ for all $M \in \mathrm{Max}(A)$ that contains J.

Proof.

I, J are both A-modules with $\phi : J \to I$. We saw that ϕ is surjective if and only if $\phi_M : J_M \to I_M$ is surjective for all $M \in \mathrm{Max}(A)$. By the previous claim, if $J \not\subseteq M$ then $J_M = A_M = I_M$. □

Recall the claim that are set after

Claim

Let A be a noetherian domain of dimension 1. TFAE:

- *A is a UFI*
- *A_M is a UFI for all $M \in \mathrm{Max}(A)$.*

Proof

In one direction, take $I \subset A_M$ nontrivial ideal. Let $j : A \hookrightarrow A_M$ be the natural inclusion. We first note that $j^{-1}(I) \subseteq M$. Indeed, we know that

$$(j^{-1}(I))_M = I$$

and that if $j^{-1}(I) \not\subseteq M$ then $(j^{-1}(I))_M = A_M$.

Proof

A UFI $\implies \exists P_1, \ldots, P_s \in \mathrm{Spec}(A)$ *s.t.*

$$P_1^{a_1} \cdots P_s^{a_s} = j^{-1}(I) \subseteq M.$$

Since $\dim(A) = 1$ *each* $P_i \in \mathrm{Max}(A)$ *and so* $M = P_1$*, say. Now,*

$$I = (j^{-1}(I))_M = \left(P_1^{a_1} \cdots P_s^{a_s}\right)_M.$$

Since product and localization commute,

$$I = (P_1 A_M)^{a_1} \cdots (P_s A_M)^{a_s}.$$

Note that

$$(P_1 A_M)^{a_1} = (M A_M)^{a_1}$$

For $i > 1$,

$$(P_i A_M)^{a_i} = A_M.$$

Proof

Thus,

$$I = (MA_M)^{a_1}.$$

Uniqueness. *Since A_M is local and $\dim(A_M) = 1$, to prove uniqueness it suffices to show that $(MA_M)^a \neq (MA_M)^{a+1}$ for every $M \in \mathrm{Max}(A)$ and $a \geq 0$. As M is the only maximal ideal of A that contains M^a, a previous claim implies that*

$$(MA_M)^a = (MA_M)^{a+1} \quad \Longleftrightarrow \quad M^a = M^{a+1}.$$

The proof then follows by UFI in A.

Proof

Now for the other direction. Take I *ideal in* A. *Let* $M_1, \ldots, M_s$ *be the (finite) set of maximal ideals in* A *containing* I. *Let* $\phi_i : A \to A_{M_i}$ *be the natural map. Since* $A_{M_1}, \ldots, A_{M_s}$ *are UFI, there exist unique integers* $a_1, \ldots, a_s \geq 1$ *s.t.*

$$\phi_i(I) = (M_i A_{M_i})^{a_i}.$$

We will now show that $I = M_1^{a_1} \cdots M_s^{a_s}$.

Proof

Note that

$$I \subseteq \phi_1^{-1}\left((M_1 A_{M_1})^{a_1}\right) \cap \cdots \cap \phi_s^{-1}\left((M_s A_{M_s})^{a_s}\right).$$

We claim that

$$\phi_i^{-1}\left((M_i A_{M_i})^{a_i}\right) = M_i^{a_i}.$$

By a previous remark, $M_i^{a_i} \subseteq \phi_i^{-1}\left((M_i A_{M_i})^{a_i}\right)$*. Since* M_i *is the only maximal ideal in* A *that contains* $M_i^{a_i}$*, it suffices to prove that*

$$(M_i A_{M_i})^{a_i} = \left(\phi_i^{-1}\left((M_i A_{M_i})^{a_i}\right)\right)_{M_i}$$

which indeed holds by a previous claim (the localization M_i *"cancels out"* ϕ_i^{-1}*). So,*

$$I \subseteq M_1^{a_1} \cap \cdots \cap M_s^{a_s}.$$

Proof

$$I \subseteq M_1^{a_1} \cap \cdots \cap M_s^{a_s}.$$

Since the $M_1, \ldots, M_s$ are pairwise coprime, so are $M_1^{a_1}, \ldots, M_s^{a_s}$, and so

$$I \subseteq M_1^{a_1} \cdots M_s^{a_s}.$$

To prove the other inclusion, it suffices to prove that for every maximal ideal M that contains I,

$$I_M = (M_1 A_M)^{a_1} \cdots (M_s A_M)^{a_s}.$$

Since M is one of the M_is,

$$(M_1 A_{M_i})^{a_1} \cdots (M_s A_{M_i})^{a_s} = (M_i A_{M_i})^{a_i} = I_{M_i}$$

Proof.

We turn to prove uniqueness. As $M_1, \ldots, M_s$ are the maximal ideals in A containing I, any factorization of I is of the form $I = M_1^{b_1} \cdots M_s^{b_s}$. Localizing at M_i, we get

$$I_{M_i} = (M_1 A_{M_i})^{b_1} \cdots (M_s A_{M_i})^{b_s} = (M_i A_{M_i})^{b_i}.$$

Similarly,

$$I_{M_i} = (M_1 A_{M_i})^{a_1} \cdots (M_s A_{M_i})^{a_s} = (M_i A_{M_i})^{a_i}.$$

By UFI of A_{M_i}, $a_i = b_i$ for all $i \in [s]$. □

Claim

Let A be a noetherian local domain of dimension 1. Then,

$$A \text{ UFI} \iff A \text{ PID}$$

Proof.

Let M be the unique maximal ideal of A. Since $\mathrm{Spec}(A) = \{\langle 0\rangle, M\}$, to prove $\implies$ it suffices to show that M is principal. Take $x \in M \setminus M^2$. By UFI of A, such x exists. Again by UFI of A and since A is a local domain of dimension 1, $\langle x\rangle = M^a$ for some integer $a \geq 1$. Since $M^a \subseteq M$ and $M^a \nsubseteq M^2$ we have $a = 1$, namely, $M = \langle x\rangle$. □

Claim

Let A be a noetherian local domain of dimension 1. Then,

$$A \text{ integrally closed} \iff A \text{ PID}$$

Proof

Let M be the unique maximal ideal of A. Since $\mathrm{Spec}(A) = \{\langle 0\rangle, M\}$, to prove $\implies$ it suffices to show that M is principal. Take $0 \neq x \in M$. Since A is noetherian of dimension 1, there exists a least integer $n \geq 1$ s.t. $M^n \subseteq \langle x\rangle$. If $n = 1$ we are done. Otherwise, take $y \in M^{n-1} \setminus \langle x\rangle$. Note that

$$\frac{y}{x} \in \mathrm{Frac}(A) \setminus A.$$

$\implies \frac{y}{x}$ *is not integral over A.*

Recall

Claim

Let A be a subring of a field L. Let $\alpha \in L$. TFAE:

1. *α is integral over A.*
2. *The subring $A[\alpha]$ of L is a f.g. A-module.*
3. *There exists a f.g. A-submodule M of L s.t. $\alpha M \subseteq M$.*

Continuing the proof

Proof

A noetherian ring $\implies$ A noetherian A-module $\implies$ M is a f.g. A-module. Since $\frac{y}{x} \in K$ and is not integral over A,

$$\frac{y}{x} M \not\subseteq M.$$

Proof.

$$\frac{y}{x}M \not\subseteq M.$$

But recall that

$$y \in M^{n-1} \implies yM \subseteq M^n \subseteq \langle x \rangle \implies \frac{y}{x}M \subseteq A$$

Therefore, $\frac{y}{x}M$ is an ideal in A not contained in M. So,

$$\frac{y}{x}M = A \implies M = \frac{y}{x}A.$$

□

Theorem

Let A be a noetherian domain of dimension 1. TFAE:

- *A is a Dedekind domain.*
- *A is a UFI*

Proof.

We ought to show that A is integrally closed $\iff$ A UFI.

- In the previous unit, we proved that integrally closed is a local property.
- We proved that UFI is a local property (in a noetherian domain of dimension 1).
- We proved that a local noetherian domain of dimension 1 is UFI $\iff$ PID $\iff$ integrally closed.

□