Introduction to Algebraic-Geometric Codes **Spring 2019**

Exercise 7

Publish Date: 02 June 19 *Due Date: 16 June 19*

Exercise 7.1. *Let A be a Dedekind domain, and let $0 \neq I, J \lhd A$. For $0 \neq P \in Spec(A)$ we define $ord_P(I)$ to be the maximal $k \in \mathbb{N}$ such that $P^k \supseteq I$, or 0 if there is no such k.*

(a) *Show that $J \subseteq I$ if and only if $ord_P(J) \geq ord_P(I)$, for all $P \in Spec(A)$.*

(b) *The greatest common divisor ideal of I and J is defined to be the ideal of A generated by I and J. Show that $(I, J) = I + J = \prod_{P \in Spec(A)} P^{\min(ord_P(I), ord_P(J))}$.*

(c) *The least common multiple ideal of I and J is defined to be the ideal $I \cap J$. Show that $I \cap J = \prod_{P \in Spec(A)} P^{\max(ord_P(I), ord_P(J))}$.*

Exercise 7.2. *Let $A = \mathbb{Z}[i] = \mathbb{Z}[x]/\left\langle x^2+1 \right\rangle$.*

(a) *Show that if $p \equiv 3 \mod 4$ then pA is a prime ideal.*

(b) *Factor $2A, 13A$ in A.*

Exercise 7.3. *In this exercise you should not use the characterization of surjective valuations of $k(x)$ over k.*

(a) *Let $g(x) \in k[x]$ be irreducible, prove that if a surjective valuation of $k(x)$ over k, v, satisfies $v(g) > 0$ then $v(g) = 1$ and $v = v_g$.*

(b) *Let L/k be an algebraic extension, prove that $\{v \mid \text{valuation of } L \text{ over } k\} = \{0\}$.*

(c) *Prove that for every irreducible polynomial $x \neq g(x) \in k[x]$ of degree d, $v_{\frac{g(x)}{x^d}} = v_{g(x)}$.*