Ordered Groups

Unit 3

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Ordered groups

Definition 1

An abelian group Γ with total order $\leq$ is an ordered group if for all $\alpha, \beta, \gamma \in \Gamma$,

$$\alpha \leq \beta \quad \Longrightarrow \quad \alpha + \gamma \leq \beta + \gamma.$$

Examples

- $\mathbb{Z}$ and $\mathbb{R}$ with the usual order.
- $\mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order.
- $\Gamma_1 \oplus \Gamma_2$ for Γ_1, Γ_2 ordered, with the lexicographic order.

Monoids

Recall that a monoid is a "group without inverses".

Definition 2 (Monoids)

A set G with a binary operation $G \times G \to G$ is a monoid if we have

- associativity: $\forall a, b, c \in G \quad (ab)c = a(bc)$; and
- identity element: $\exists e \in G \;\; \forall a \in G \quad ae = ea = a$.

$(\mathbb{N}, +)$ and $(\mathbb{N} \setminus \{0\}, \cdot)$ are notable examples of monoids.

As usual, $H \subseteq G$ is a submonoid of G if the operation inherited from G is closed in H. That is, $\forall a, b \in H$ it holds that $ab \in H$. We further require that the unit of H is the unit of G.

A monoid G is abelian if $ab = ba$ for all $a, b \in G$.

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Claim 3

Let Γ be an ordered group. Then,

1. $\gamma \geq 0 \implies -\gamma \leq 0$.
2. $\Gamma_+ = \{\gamma \in \Gamma \mid \gamma \geq 0\}$ is a submonoid of Γ.
3. $\Gamma_+ \cap (-\Gamma_+) = \{0\}$.
4. $\Gamma_+ \cup (-\Gamma_+) = \Gamma$.

The proof is straightforward and is left as an exercise.

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A kind of an inverse to Claim 3 holds as well.

Claim 4

Let Γ be an abelian group with a submonoid Γ_+ satisfying

$$\Gamma_+ \cap (-\Gamma_+) = \{0\},$$
$$\Gamma_+ \cup (-\Gamma_+) = \Gamma.$$

Define an order on Γ by

$$\alpha \leq \beta \iff \beta - \alpha \in \Gamma_+.$$

Under this order, Γ is an ordered group.

The proof is straightforward and is left as an exercise.

As an example, the natural order of $\mathbb{Z}$ is determined by $\mathbb{N}$.

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Claim 5

Let Γ be an ordered group. For $n \in \mathbb{N}$ define $\varphi_n : \Gamma \to \Gamma$ mapping $\gamma \mapsto n\gamma$. Then, φ_n is an ordered preserving monomorphism.

Proof.

Γ abelian $\implies$ φ_n is a group homomorphism. Indeed,

$$\varphi_n(\alpha + \beta) = n(\alpha + \beta) = n\alpha + n\beta = \varphi_n(\alpha) + \varphi_n(\beta).$$

Take $\alpha \in \ker \varphi_n$ and assume wlog $\alpha \geq 0$. By induction on n, one can prove that $n\alpha \geq \alpha$, and so

$$0 = \varphi_n(\alpha) = n\alpha \geq \alpha \geq 0 \quad \implies \quad \alpha = 0.$$

Thus, $\ker \varphi_n = 0$, and so φ_n is a group monomorphism.

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Proof.

We are left to show that φ_n is order preserving.

Take $\alpha \leq \beta$. We prove by induction on $n \geq 1$ that $n\alpha \leq n\beta$. Assuming for $n-1$, we get

$$n\alpha = (n-1)\alpha + \alpha \leq (n-1)\beta + \beta = n\beta.$$

□

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Corollary 6

Let Γ be an ordered group. Then, every $0 \neq \gamma \in \Gamma$ has infinite order. In particular, unless $\Gamma = \{0\}$, Γ is infinite.

Proof.

If $o(\gamma) = n < \infty$ then $\gamma \in \ker \varphi_n$. However, by Claim 5, $\ker \varphi_n = \{0\}$, thus $\gamma = 0$. □

Extending an ordered group by adjoining ∞

Let Γ be an ordered group. It is convenient to adjoin to Γ an element $\infty > \Gamma$, and define

$$\gamma + \infty = \infty + \gamma = \infty + \infty = \infty.$$

Note that $\Gamma \cup \{\infty\}$ is no longer a group.