Spectral Graph Theory **Winter 2020**

Problem Set 5

Publish Date: December 29, 2020 *Due Date: January 12, 2021*

Problem 5.1 Let $G = (V, E, w)$ be a graph, prove the following statements:

1. $\sum_{a,b \in V} R_{\text{eff}}(a, b) = n \sum \frac{1}{\lambda_i}$ where we sum over non zero eigenvalues of $L(G)$.
2. G is a tree $\iff$ for every $a, b \in V$, $R_{\text{eff}}(a, b) = \sum_{e \in \delta} \frac{1}{w_e}$ where δ is the lightest path between a and b.

Problem 5.2

1. Let M be a PSD, and let X be a non zero, diagonal matrix with non negative entries. Prove that $M + X$ is positive definite.
2. Let $G = (V, E, w)$ be a graph, and let $\emptyset \neq B \subseteq V$. Denote by $S = V \setminus B$. Prove that $L(S, S) = L(G_S) + X$ where $L(S, S)$ is the sub-matrix of $L(G)$ defined by taking the rows and columns of S, and X is a diagonal matrix such that $X_{a,a} = \sum_{b \in B, (a,b) \in E} w_{a,b}$.
3. Conclude that for every graph $G = (V, E, w)$, a subset $B \subseteq V$ and a function $f : B \to \mathbb{R}$ there is a unique extension $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}|_B = f$ and $\tilde{f}$ is harmonic on $V \setminus B$.

Problem 5.3 In class we analyzed the width-3 replacement product when operating on error vectors of the form $\mathbf{y} \otimes (\mathbf{u} \otimes \mathbf{u})$ where $\mathbf{y} \perp \mathbf{1}$. Complete the analysis of the width-3 replacement product for general error vectors $\mathbf{x} \perp \mathbf{1}$.