Riemann's Theorem and the Genus

Unit 11

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December 3, 2024

Overview

1. Principal, zero, and pole divisors
2. The group of principal divisors and the divisor class group
3. The degree of the zero and pole divisors
4. Riemann's Theorem
5. The Genus

Principal, zero, and pole divisors

Definition 1

Let F/K be a function field. For an element $x \in F^\times$ we defined the principal divisor of x by

$$(x) = \sum_{\mathfrak{p} \in \mathbb{P}} v_\mathfrak{p}(x)\mathfrak{p} \in \widetilde{\mathcal{D}}.$$

We further define the zero divisor and pole divisor of x by

$$(x)_0 = \sum_{\substack{\mathfrak{p} \in \mathbb{P} \\ v_\mathfrak{p}(x) > 0}} v_\mathfrak{p}(x)\mathfrak{p} \in \widetilde{\mathcal{D}},$$

$$(x)_\infty = -\sum_{\substack{\mathfrak{p} \in \mathbb{P} \\ v_\mathfrak{p}(x) < 0}} v_\mathfrak{p}(x)\mathfrak{p} \in \widetilde{\mathcal{D}}.$$

We will soon justify the name divisor of these pseudo divisors.

Principal, zero, and pole divisors

Lemma 2

Let F/K *be a function field. Let* $x \in F^\times$ *and* $S \subseteq \mathbb{P}$ *finite s.t.*

$$\forall \mathfrak{p} \in S \quad v_{\mathfrak{p}}(x) > 0.$$

Then,

$$\deg (x)_S \le [F : K(x)].$$

Proof.

The assertion is trivial for $x \in K^\times$, so assume $x \in F \setminus K$. Let

$$\mathfrak{a} = (x)_S = \sum_{\mathfrak{p} \in S} v_{\mathfrak{p}}(x)\mathfrak{p} \ge 0.$$

By a result we proved,

$$\dim_K \mathcal{L}(\mathfrak{a}, S) \Big/ \mathcal{L}(0, S) = \deg \mathfrak{a}_S - \deg 0_S = \deg \mathfrak{a}.$$

Principal, zero, and pole divisors

Proof.

$$\dim_{\mathrm{K}} \mathcal{L}(\mathfrak{a}, S) \Big/ \mathcal{L}(0, S) = \deg \mathfrak{a}.$$

So we interpreted $\deg \mathfrak{a}$ as a dimension of a certain K-vector space.

Hence, it suffices to prove that for any $k > [\mathrm{F} : \mathrm{K}(x)]$, any $y_1, \ldots, y_k \in \mathcal{L}(\mathfrak{a}, S)$ are linearly dependent over K modulo $\mathcal{L}(0, S)$.

That is, we want to find $a_1, \ldots, a_k \in \mathrm{K}$, not all zeros, s.t.

$$\sum_{i=1}^{k} a_i y_i \in \mathcal{L}(0, S).$$

As $k > [\mathrm{F} : \mathrm{K}(x)]$ there are $f_1(x), \ldots, f_k(x) \in \mathrm{K}(x)$, not all zeros, s.t.

$$\sum_{i=1}^{k} f_i(x) y_i = 0.$$

Principal, zero, and pole divisors

Proof.

There are $f_1(x), \ldots, f_k(x) \in \mathrm{K}(x)$, not all zeros, s.t.

$$\sum_{i=1}^{k} f_i(x) y_i = 0.$$

We may assume all $f_i(x) \in \mathrm{K}[x]$, and not all are divisible by x in $\mathrm{K}[x]$.

Write

$$f_i(x) = g_i(x) + a_i, \quad g_i(x) \in x\mathrm{K}[x],\ a_i \in \mathrm{K}.$$

Thus,

$$\sum_{i=1}^{k} a_i y_i = -\sum_{i=1}^{k} g_i(x) y_i.$$

Note that not all a_i's are zero. So it suffices to show that RHS $\in \mathcal{L}(0, S)$.

Principal, zero, and pole divisors

Proof.

We wish to show that

$$\sum_{i=1}^{k} g_i(x) y_i \in \mathcal{L}(0, S).$$

To this end, it suffices to show that

$$0 \neq g(x) \in x\mathsf{K}[x],\ \ y \in \mathcal{L}(\mathfrak{a}, S) \quad \Longrightarrow \quad g(x)y \in \mathcal{L}(0, S).$$

Fix $\mathfrak{p} \in S$, and note that, as $\upsilon_{\mathfrak{p}}(x) > 0$,

$$\upsilon_{\mathfrak{p}}(g(x)) \geq \upsilon_{\mathfrak{p}}(x).$$

Therefore, $(g(x))_S \geq (x)_S = \mathfrak{a}$, and so

$$\begin{aligned} g(x)y \in g(x)\mathcal{L}(\mathfrak{a}, S) &= \mathcal{L}(\mathfrak{a} - (g(x)), S) \\ &= \mathcal{L}(\mathfrak{a} - (g(x))_S, S) \subseteq \mathcal{L}(0, S). \end{aligned}$$

□

Principal, zero, and pole divisors

Corollary 3

For all $x \notin \mathsf{F}^\times$,

$$(x), (x)_0, (x)_\infty \in \mathcal{D}.$$

Moreover, if $x \in \mathsf{F} \setminus \mathsf{K}$ *then*

$$\deg(x)_0, \deg(x)_\infty \leq [\mathsf{F} : \mathsf{K}(x)].$$

Proof.

The proof is straightforward for $x \in \mathsf{K}^\times$, so assume $x \in \mathsf{F} \setminus \mathsf{K}$.

Lemma 2 implies that $(x)_0$ must be a divisor as the number of places $\mathfrak{p}$ that appear in $(x)_0$ cannot exceed $[\mathsf{F} : \mathsf{K}(x)] < \infty$. Indeed,

$$\deg(x)_0 \leq [\mathsf{F} : \mathsf{K}(x)].$$

To prove the assertion regarding $(x)_\infty$, recall it is equal to $(x^{-1})_0$. □

Overview

The group of principal divisors and the divisor class group

Definition 4

The set of all principal divisors of F/K is called the group of principal divisors of F/K

$$\mathcal{P} = \left\{ (x) \mid x \in \mathsf{F}^{\times} \right\}.$$

$\mathcal{P}$ is indeed a group as

$$(x) + (y) = (xy)$$
$$(1) = 0$$
$$(x) + (x^{-1}) = (xx^{-1}) = (1) = 0.$$

$\mathcal{P}$ is a subgroup of $\mathcal{D}$.

Definition 5

The group $\mathcal{C} = \mathcal{D}/\mathcal{P}$ is called the divisor class group.

Overview

The degree of the zero and pole divisors

Definition 6 (Integral elements)

Let S/R be a ring extension. An element $x \in S$ is said to be integral over R if there exists a monic $f(T) \in \mathrm{R}[T]$ s.t. $f(x) = 0$.

Claim 7

Let F/K be a function field. Let $x \in \mathrm{F} \setminus \mathrm{K}$ and $y_1, \ldots, y_n \in \mathrm{F}$. Then,

1. If $y_1, \ldots, y_n$ are linearly independent over $\mathrm{K}(x)$ then
$$\{x^i y_j \mid i \geq 0, j \in [n]\}$$
are linearly independent over K.
2. There exists $0 \neq g(x) \in \mathrm{K}[x]$ s.t. $g(x)y_1, \ldots, g(x)y_n \in \mathrm{F}$ are integral over $\mathrm{K}[x]$.

The degree of the zero and pole divisors

Proof.

The first item is straightforward and is left as an exercise.

Consider any fixed $y = y_j$.

As $[\mathrm{F} : \mathrm{K}(x)] < \infty$, $\mathrm{F}/\mathrm{K}(x)$ is algebraic. Hence, $\exists f_0, \ldots, f_{d-1} \in \mathrm{K}(x)$, not all zero, s.t.

$$y^d + f_{d-1}y^{d-1} + \cdots + f_1 y + f_0 = 0.$$

So, for an appropriate choice of $g \in \mathrm{K}[x]$, we get

$$(gy)^d + (gf_{d-1})(gy)^{d-1} + \cdots + g^d f_0 = 0,$$

with $gf_i \in \mathrm{K}[x]$. Thus, gy is integral over $\mathrm{K}[x]$.

The same argument can be extended to all of $y_1, \ldots, y_n$ simultaneously. □

The degree of the zero and pole divisors

Claim 8

Let F/K be a function field. Let $x \in F$, and let $y \in F$ integral over $K[x]$. Then for every $\mathfrak{p} \in \mathbb{P}$,

$$\upsilon_{\mathfrak{p}}(x) \geq 0 \quad \Longrightarrow \quad \upsilon_{\mathfrak{p}}(y) \geq 0.$$

Proof.

Take $f_0(x), \ldots, f_{d-1}(x) \in K[x]$ s.t.

$$y^d + f_{d-1}(x)y^{d-1} + \cdots + f_1(x)y + f_0(x) = 0.$$

We may assume $y \neq 0$ and write

$$y = -f_{d-1}(x) - f_{d-2}(x)y^{-1} - \cdots - f_0(x)(y^{-1})^{d-1}.$$

The degree of the zero and pole divisors

Proof.

$$y = -f_{d-1}(x) - f_{d-2}(x)y^{-1} - \cdots - f_0(x)(y^{-1})^{d-1}.$$

As $v_{\mathfrak{p}}(x) \geq 0$ we have that $v_{\mathfrak{p}}(f_i(x)) \geq 0$.

Had it been the case that $v_{\mathfrak{p}}(y) < 0$ we would get $v_{\mathfrak{p}}(y^{-1}) > 0$ and so, $v_{\mathfrak{p}}(\text{RHS}) \geq 0$, contradicting the assumption $v_{\mathfrak{p}}(y) < 0$.

The degree of the zero and pole divisors

Theorem 9

Let $x \in \mathsf{F} \setminus \mathsf{K}$*. Then,*

$$\deg(x)_0 = \deg(x)_\infty = [\mathsf{F} : \mathsf{K}(x)].$$

In particular, $\deg(x) = 0$*.*

The theorem, in particular, says that every function has the same number of zeros and poles, when counted with multiplicities.

Proof.

It suffice to prove that $\deg(x)_\infty = [\mathsf{F} : \mathsf{K}(x)]$ as $(x)_0 = (x^{-1})_\infty$, and since

$$[\mathsf{F} : \mathsf{K}(x)] = [\mathsf{F} : \mathsf{K}(x^{-1})].$$

Moreover, by Corollary 3, it suffices to prove that

$$\deg(x)_\infty \geq [\mathsf{F} : \mathsf{K}(x)].$$

The degree of the zero and pole divisors

Proof.

We wish to prove that

$$\deg(x)_\infty \geq [\mathsf{F} : \mathsf{K}(x)] = n.$$

Take $y_1, \ldots, y_n \in \mathsf{F}$ that are linearly independent over $\mathsf{K}(x)$. By Claim 7, we may assume these are integral over $\mathsf{K}[x]$.

Claim 8 implies that if $v_\mathfrak{p}(y_j) < 0$ for some $j \in [n]$ then $v_\mathfrak{p}(x) < 0$.

Since Corollary 3 implies that $(x)_\infty$ is supported on finitely many prime divisors, for a sufficiently large integer k it holds that

$$k(x)_\infty \geq (y_j)_\infty \quad \forall j \in [n].$$

The degree of the zero and pole divisors

Proof.

Now, for any integer $\ell \geq 0$ we have that for every $0 \leq i \leq \ell$,

$$\begin{aligned}(x^i y_j) + (k+\ell)(x)_\infty &= i(x) + (y_j) + k(x)_\infty + \ell(x)_\infty \\ &= i(x)_0 - i(x)_\infty + (y_j)_0 - (y_j)_\infty + k(x)_\infty + \ell(x)_\infty \\ &= i(x)_0 + (y_j)_0 + (k(x)_\infty - (y_j)_\infty) + (\ell - i)(x)_\infty \\ &\geq 0.\end{aligned}$$

Thus,

$$\{x^i y_j \mid 0 \leq i \leq \ell,\ j \in [n]\} \subseteq \mathcal{L}((k+\ell)(x)_\infty).$$

As by Claim 7, the above are linearly independent over K,

$$\dim(k+\ell)(x)_\infty \geq n(\ell+1).$$

The degree of the zero and pole divisors

Proof.

$$\dim(k+\ell)(x)_\infty \geq n(\ell+1). \tag{1}$$

Recall though that we proved that for every positive divisor $\mathfrak{a} \geq 0$,

$$\dim \mathfrak{a} \leq \deg \mathfrak{a} + 1.$$

Thus,

$$(k+\ell)\deg(x)_\infty = \deg(k+\ell)(x)_\infty \geq n(\ell+1) - 1,$$

and so, as $\ell \to \infty$,

$$\deg(x)_\infty \geq \frac{\ell+1}{\ell+k} \cdot n - \frac{1}{\ell+k} \longrightarrow n,$$

implying $\deg(x)_\infty \geq n$.

The degree of the zero and pole divisors

By inspecting the proof of Theorem 9 we also conclude

Corollary 10

$\forall x \in \mathsf{F} \setminus \mathsf{K} \quad \exists q \in \mathbb{N}$ *s.t.*

$$\forall m \in \mathbb{N} \quad \deg m(x)_\infty - \dim m(x)_\infty \leq q.$$

Proof.

In Equation (1) we showed that $\exists k \in \mathbb{N}$ s.t. $\forall \ell \geq 0$

$$\dim(k+\ell)(x)_\infty \geq (\ell+1)\deg(x)_\infty.$$

Write $m = k + \ell$. Then, $\forall m \geq k$,

$$\dim m(x)_\infty \geq (m-k+1)\deg(x)_\infty.$$

The degree of the zero and pole divisors

Proof.

Equivalently, $\forall m \geq k$,

$$\deg m(x)_\infty - \dim m(x)_\infty \leq (k-1)\deg(x)_\infty = q.$$

Note that q is independent of m.

The proof then follows by the monotonicity of $\deg - \dim$.

Overview

Riemann's Theorem

Theorem 11 (Riemann's Theorem)

Let F/K *be a function field, and* $x \in F \setminus K$. *Define*

$$q = \max\left\{\deg m(x)_\infty - \dim m(x)_\infty \;\middle|\; m \in \mathbb{N}\right\}.$$

Then,

$$\forall \mathfrak{a} \in \mathcal{D}_{F/K} \quad \deg \mathfrak{a} - \dim \mathfrak{a} \leq q.$$

Proof.

Recall that for divisors $\mathfrak{a}, \mathfrak{b}$,

$$\mathfrak{a} \leq \mathfrak{b} \quad \Longrightarrow \quad \deg \mathfrak{a} - \dim \mathfrak{a} \leq \deg \mathfrak{b} - \dim \mathfrak{b}, \tag{2}$$

and so we may assume $\mathfrak{a} \geq 0$.

Riemann's Theorem

Proof.

For every $m \in \mathbb{N}$,

$$m(x)_\infty - \mathfrak{a} \leq m(x)_\infty,$$

and so by invoking Equation (2) again

$$\deg(m(x)_\infty - \mathfrak{a}) - \dim(m(x)_\infty - \mathfrak{a}) \leq \deg m(x)_\infty - \dim m(x)_\infty \leq q.$$

Therefore,

$$\begin{aligned} \dim(m(x)_\infty - \mathfrak{a}) &\geq \deg(m(x)_\infty - \mathfrak{a}) - q \\ &= m \deg(x)_\infty - \deg \mathfrak{a} - q. \end{aligned}$$

Thus, for a sufficiently large m, $\dim(m(x)_\infty - \mathfrak{a}) > 0$ and we can find

$$0 \neq y \in \mathcal{L}(m(x)_\infty - \mathfrak{a}).$$

Riemann's Theorem

Proof.

$$0 \neq y \in \mathcal{L}(m(x)_\infty - \mathfrak{a}).$$

Thus,

$$(y) + m(x)_\infty - \mathfrak{a} \geq 0,$$

equivalently,

$$\mathfrak{a} + (y^{-1}) \leq m(x)_\infty.$$

Invoking Equation (2) again we get

$$\deg\left(\mathfrak{a} + (y^{-1})\right) - \dim\left(\mathfrak{a} + (y^{-1})\right) \leq \deg\left(m(x)_\infty\right) - \dim\left(m(x)_\infty\right) \leq q.$$

The proof then follows as

$$\deg\left(\mathfrak{a} + (y^{-1})\right) = \deg \mathfrak{a} + \deg(y^{-1}) = \deg \mathfrak{a},$$
$$\dim\left(\mathfrak{a} + (y^{-1})\right) = \dim \mathfrak{a}.$$

Overview

The Genus

Definition 12 (The Genus)

Let F/K be a function field. The number g satisfying

$$g - 1 = \max\left\{\deg \mathfrak{a} - \dim \mathfrak{a} \mid \mathfrak{a} \in \mathcal{D}_{F/K}\right\}$$

is called the genus of F/K.

Note that $g \geq 0$. Indeed, $\mathcal{L}(0) = K$ and so $\dim 0 = 1$. Thus,

$$g - 1 \geq \deg 0 - \dim 0 = 0 - 1.$$

The Genus

Observe that $\forall x \in \mathsf{F} \setminus \mathsf{K}$,

$$g - 1 = \max\left\{\deg m(x)_\infty - \dim m(x)_\infty \mid m \in \mathbb{N}\right\}.$$

Indeed, the RHS was defined to be $q = q_x$ with respect to a specific x. But then by Riemann's Theorem,

$$q_y = \max\left\{\deg m(y)_\infty - \dim m(y)_\infty \mid m \in \mathbb{N}\right\} \le q_x.$$

As the argument works for all x, y we get $q_x = q_y$.

The genus of the rational function field

Claim 13

The genus of the rational function field $K(x)/K$ is 0.

Proof.

By the above remark,

$$g - 1 = \max\left\{\deg m(x)_\infty - \dim m(x)_\infty \;\middle|\; m \in \mathbb{N}\right\}$$

Since

$$(x) = \mathfrak{p}_0 - \mathfrak{p}_\infty,$$

we have that

$$\begin{aligned} \deg m(x)_\infty &= m \deg (x)_\infty = m \deg \mathfrak{p}_\infty = m, \\ \dim m(x)_\infty &= \dim m\mathfrak{p}_\infty = m + 1, \end{aligned}$$

and so

$$g - 1 = \max_{m \in \mathbb{N}} (m - (m+1)) \quad \Longrightarrow \quad g = 0.$$

Exercise

Recall that we proved that for every divisor $\mathfrak{a} \geq 0$,

$$\dim \mathfrak{a} \leq \deg \mathfrak{a} + 1.$$

Exercise. Prove that the bound holds for all $\mathfrak{a} \in \mathcal{D}$ with $\deg \mathfrak{a} \geq 0$.

Clifford's Theorem

We will later see that

$$\deg \mathfrak{a} \geq 2g - 1 \quad \Longrightarrow \quad \dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g.$$

In the assignment you will prove a result on the lower degree divisors.

Theorem 14 (Clifford's Theorem)

$\forall \mathfrak{a} \in \mathcal{D}$ *with* $0 \leq \deg \mathfrak{a} \leq 2g - 2$,

$$\dim \mathfrak{a} \leq 1 + \frac{1}{2} \cdot \deg \mathfrak{a}.$$

The proof is based on The Riemann-Roch Theorem and on

Lemma 15

$\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{D}$ *with* $\dim \mathfrak{a}, \dim \mathfrak{b} > 0$,

$$\dim \mathfrak{a} + \dim \mathfrak{b} \leq 1 + \dim(\mathfrak{a} + \mathfrak{b}).$$