Algebraic Geometric Codes

Recitation 08

Shir Peleg

Tel Aviv University

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Degree 2 extensions function fields.

Let $F/K(x)$ be an algebraic extension of with $[F : K(x)] = 2$. Assume further thet $char(K) \neq 2$. We want to compute the genus and Riemann Roch spaces of some divisors.

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Claim 0.1

There is $y \in F$ such that $F = K(y, x)$, and $y^2 = d(x)$, where $d(x) \in F[x]$, and all it's factors has multiplicity 1.

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Set $y_2 = y_1 + b/2$ s.t. $y_2^2 = b^2/4 - c$, and is a generator of $F/K(x)$.

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Computing the Genus

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Denote $m = \deg d$. *It holds that for every* $n \in \mathbb{N}$,
$\dim \mathcal{L}(n(x)_\infty) = \lfloor 2n + 2 - \frac{m}{2} \rfloor = 2n + 2 - \lceil \frac{m}{2} \rceil$

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write $z = f(x) + y \cdot g(x)$, $f, g \in K(x)$, $\sigma(z) = f(x) - y \cdot g(x)$.

□

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Continue.

$z + \sigma(z) = 2f(x)$, $z\sigma(z) = f^2(x) - d(x)g^2(x)$. As both $z, \sigma(z) \in \mathcal{L}(n(x)_\infty)$, it follows that

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$z+\sigma(z)=2f(x)\in\mathcal{L}(n(x)_\infty)\cap K(x)\rightarrow f\in K[x]$ and $\deg f\leq n$.

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$\to f^2-d(x)g^2(x)\in K[x]$ and $\deg(f^2-d(x)g^2(x))\leq 2n$

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It follows that $dg^2 \in K[x]$, and a d does not have multiple factors, thus $g \in K[x]$. $\deg(g) \leq \frac{2n-m}{2} = n - \frac{m}{2}$.

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We conclude that $z = f(x) + y \cdot g(x)$, where $f, g \in K[x]$, and $\deg(f) \leq n$, $\deg g \leq n - \frac{m}{2}$.

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It follows that $dg^2 \in K[x]$, and a d does not have multiple factors, thus $g \in K[x]$. $\deg(g) \leq \frac{2n-m}{2} = n - \frac{m}{2}$.
We conclude that $z = f(x) + y \cdot g(x)$, where $f, g \in K[x]$, and $\deg(f) \leq n$, $\deg g \leq n - \frac{m}{2}$.
In the other direction, let $z = f + y \cdot g$ where f, g as above, we will show that $z \in \mathcal{L}(n(x)_\infty)$.

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In the other direction, let $z = f + y \cdot g$ where f, g as above, we will show that $z \in \mathcal{L}(n(x)_\infty)$.

Indeed, $(x) \geq -(x)_\infty$. It holds that $(y^2) = (d(x)) \geq -m(x)_\infty$, therefore $(y) \geq \frac{m}{2}(x)_\infty$.

It follows that $(yg(x)) = (y) + (g(x)) \geq -\frac{m}{2}(x)_\infty - (\deg g)(x)_\infty \geq -n(x)_\infty$ and $(f(x)) \geq -(\deg f)(x)_\infty \geq n(x)_\infty$ and thus $f(x), yg(x) \in \mathcal{L}(n(x)_\infty)$, and so is z, and the claim follows. □

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To conclude, we showed that $\mathcal{L}(n(x)_\infty) = span\{x^i, yx^j\}_{i \leq n, j \leq n - \frac{m}{2}}$, and thus $\dim(\mathcal{L}(n(x)_\infty)) = 2n + 2 - \frac{m}{2}$.
Thus, for large enough $n \in \mathbb{N}$, s.t $n \geq 2g - 2$, and thus for a Canonical divisor K, $\deg(K - n(x)_\infty) = \deg(K) - n\deg(x)_\infty = \deg(K) - 2n < 0$ as $\deg(x)_\infty = [F : K(x)] = 2$.

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Therefore, from R.R theorem, $\dim(\mathcal{L}(()n(x)_\infty)) = 2n - g + 1$, and thus:

$$2n + 2 - \lceil \frac{m}{2} \rceil = 2n - g + 1 \Rightarrow g = \lceil \frac{m}{2} \rceil - 1$$