Recap from Before Passover

Introduction to Algebraic-Geometric Codes. Fall 2019.

April 27, 2019

For your convenience, here is a detailed recap of what we did in the last lecture.

Definition

Let K, L be fields. An embedding of K in L is a ring homomorphism $\sigma : K \to L$.

Remark

Any embedding is, in fact, a monomorphism $\sigma : K \hookrightarrow L$.

Definition

Let K, L be fields containing a field F. An embedding of K in L over F is an embedding $\sigma : K \hookrightarrow L$ such that $\sigma_{|F} = \mathrm{id}_F$.

Theorem (Steinitz's Theorem)

Let F be a field and $\bar{F}$ a fixed choice of an algebraic closure of F. Then, for every algebraic extension K/F there exists an embedding $\sigma : K \hookrightarrow \bar{F}$ over F.

Theorem (Steinitz's Theorem 2.0)

Let $F, \bar{F}$ as above. Let $K \subseteq L$ be two algebraic extensions of F. Then, for every embedding $\sigma : K \hookrightarrow \bar{F}$ over F there exists an embedding $\tau : L \hookrightarrow \bar{F}$ (over F) such that $\tau_{|K} = \sigma$. We call τ an extension of σ.

Definition

Let $F, K, \bar{F}$ as above. We define

$$\Gamma_{K/F} = \left\{\sigma : K \hookrightarrow \bar{F} \mid \sigma \text{ embedding over } F\right\}$$

We further define $\Gamma_F = \Gamma_{\bar{F}/F}$.

Claim

The elements in Γ_F are automorphisms. Hence, Γ_F has a group structure w.r.t. composition.

Definition

Let K/F be an algebraic extension. $\alpha, \beta \in K$ are conjugates over F if they share their min poly over F.

Claim

Let K/F be an algebraic extension. Let $\alpha, \beta \in K$. Then,

$$\alpha, \beta \text{ conjugates over } F \iff \exists \sigma \in \Gamma_{K/F} \ \sigma(\alpha) = \beta.$$

Proof sktech.

$\Leftarrow$ easy.
$\Rightarrow$ follows since $F(\alpha) \cong F[y]/\langle f(y) \rangle \cong F(\beta)$, where f is the shared min poly. □

Now this claim we had problems with last time, so let's do it right.

Claim

Let K be a field with algebraic closure $\bar{K}$. Let $\alpha \in \bar{K}$ separable over K. Then,

$$\alpha \in K \iff \forall \sigma \in \Gamma_K \quad \sigma(\alpha) = \alpha.$$

Proof.

$\Rightarrow$ is trivial. For $\Leftarrow$, consider α's min poly $f(y) \in K[y]$ over K. f cannot have a root $\beta \neq \alpha$ by the previous claim. So, $f(y) = (y - \alpha)^n$. Since α is separable, $f(y) = y - \alpha$ and so $\alpha \in K$. □

Definition (Noetherian ring)

A ring is noetherian if each of its ideal is finitely generated.

Definition (Noetherian module)

An A-module M is noetherian if every A-submodule of M is finitely generated.

Remark

Let A be a ring. Then,

$$A \text{ noetherian ring} \iff A \text{ noetherian } A\text{-module.}$$

This is because the A-submodules of the A-module A are precisely the ideals of the ring A.

Lemma

Let A be a noetherian ring. Let M be a f.g. A-module. Then, M is a noetherian A-module.

Corollary

Let $A \subseteq B$ be rings. Assume that A is a noetherian ring, and B is a f.g. A-module. Then, B is a noetherian ring.