Introduction

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Overview

What this course is about?

Welcome to SGT 2020/21!

Spectral graph theory is the study of graphs by their eigenvalues and eigenvectors.

- A beautiful, intuitive, and active research area.
- Highly applicable, in particular to TCS.
- Not a crazily-difficult theory to grasp (this is intended mostly for those who took my AG codes course :)).
- Many great resources.

Tentative syllabus

We plan on covering

1. The basic linear algebra we need.
2. Gain intuition on the combinatorics hinted by eigenvalues and eigenvectors. E.g., via graph drawing and a Physics-oriented point of view.
3. Combinatorial results: coloring, Paley graphs are Ramsey graphs.
4. Expander graphs - properties, constructions, and applications.
5. Reingold's famous $\mathbf{SL} = \mathbf{L}$.
6. Ta-Shma's explicit codes close to the Gilbert-Varshamov bound.
7. Graph sparsification.

Course mechanics

1. Lectures on Tuesday 9:10-12:00 via Zoom.
2. Shir's recitation on Tuesday 12:10-13:00 via Zoom.
3. Course homepage:
   `www.gilcohen.org/2020-spectral-graph-theory`.
   Everything but grades will be there.
4. Lectures will combine slides with "board" (iPad).
5. Theoretical course but I will use code to give insight.
   Following Spielman, I'm using Julia via Jupyter notebooks.
6. About 5 problem sets to be submitted in pairs will determine half of the grade. Remaining half is based on a take-home one day exam (individual submissions).
7. We will mostly follow Spielman's great book-in-progress and Vadhan's monograph on pseudorandomness.

This is a graduate-level course

1. May require Spontaneous prerequisite catching up.
2. Some problems in the problems sets and final will be difficult.
3. Fluidic course structure.
4. PS. First offering of the course.

Graphs and their matrices

Unless otherwise stated, our graphs will be undirected simple graphs, sometimes weighted. Always finite.

We will use matrices in two different ways:

- as an operator $\mathbf{x} \mapsto \mathbf{Mx}$; and
- defining a quadratic form $\mathbf{x} \mapsto \mathbf{x}^T\mathbf{Mx} = \sum_{i,j} \mathbf{M}_{i,j}\mathbf{x}_i\mathbf{x}_j$.

The adjacency matrix

Perhaps the most natural matrix associated to a graph G is its adjacency matrix $\mathbf{M}_G$ given by

$$\mathbf{M}_G(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

We index the rows and columns of $\mathbf{M}_G$ by V.

The path graph

0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0
0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	1	0

```
function pathgraph(n)
    M = [ (j==i+1 || j==i-1) ? 1 : 0 for i in 1:n, j in 1:n]
end

display(pathgraph(10))
```

The diffusion operator

The most natural operator associated with a graph G is probably its diffusion operator.

Let $\mathbf{D}_G$ be the diagonal matrix with $\mathbf{D}_G(v, v) = \deg v$. For weighted graphs, $\deg v$ is the sum of weights over edges incident to v.

Assuming there are no isolated vertices, $\mathbf{D}_G$ is invertible and we define the diffusion operator by

$$\mathbf{W}_G = \mathbf{M}_G \mathbf{D}_G^{-1}.$$

When G is d-regular then $\mathbf{D}_G = d\mathcal{I}$ and so $\mathbf{W}_G = \frac{1}{d}\mathbf{M}_G$.

The diffusion operator

Think of a vector $\mathbf{p} \in \mathbb{R}$ specifying how much stuff there is at each vertex. After one time step, the distribution of stuff is $\mathbf{W}_G\mathbf{p}$.

One important example is when $\mathbf{p}$ is a probability distribution. The operator $\mathbf{W}_G$ then captures taking a random step. Sometime we will be interested in a *lazy random walk*

$$\widetilde{\mathbf{W}}_G = \frac{1}{2}\mathcal{I} + \frac{1}{2}\mathbf{W}_G.$$

The Laplacian

The Laplacian $\mathbf{L}_G = \mathbf{D}_G - \mathbf{M}_G$ yields the most natural quadratic form associated with a graph.

$$\mathbf{x}^T \mathbf{L}_G \mathbf{x} = \sum_{uv \in E} (\mathbf{x}(u) - \mathbf{x}(v))^2.$$

This measures the "smoothness" of $\mathbf{x}$ across the edges of G.

Eigenvalues and eigenvectors

Recall that a nonzero vector ψ is an eigenvector of a matrix $\mathbf{M}$ with eigenvalue λ if

$$\mathbf{M}\psi = \lambda\psi.$$

Equivalently,

- $\lambda\mathcal{I} - \mathbf{M}$ is singular;
- λ is a root of the characteristic polynomial of $\mathbf{M}$, $\det(x\mathcal{I} - \mathbf{M})$.

Eigenvalues and eigenvectors

W= Let G be a graph. We think of a vector as a map $\psi : V \to \mathbb{R}$.

ψ is an eigenvector of $\mathbf{M}_G$ with eigenvalue λ iff for every vertex $v \in V$

$$\sum_{uv \in E} \psi(u) = \lambda \cdot \psi(v).$$

Every real (or even complex) matrix **M** has between 1 to n eigenvalues in $\mathbb{C}$. Eigenvalues and eigenvectors are suitable for studying symmetric matrices (or matrices similar to symmetric matrices). Otherwise, one should consider singular values and singular vectors.

Theorem (The Spectral Theorem)

Let **M** *be an* $n \times n$ *real, symmetric matrix. Then there exist* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *(not necessarily distinct) and n mutually orthogonal unit vectors* $\psi_1, \ldots, \psi_n$ *such that* ψ_i *is an eigenvector of* **M** *of eigenvalue* λ_i*.*

Eigenvalues and their graphs

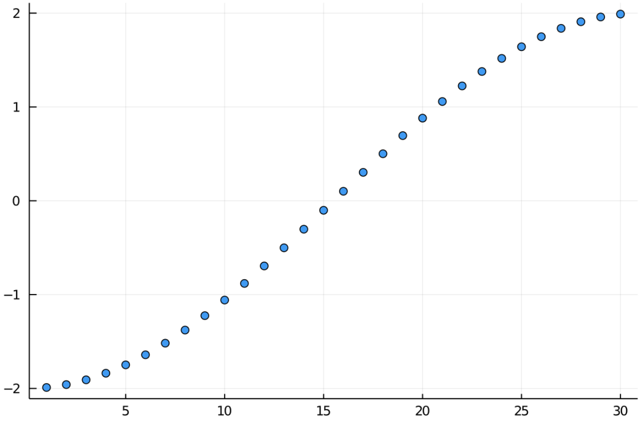


Figure: Eigenvalues of the adjacency matrix **M** of the 30-vertex path graph (largest is $\lambda_{30} \sim 1.989$).

Eigenvalues and their graphs

```
using LinearAlgebra
using Plots

function pathgraph(n)
    M = [ (j==i+1 || j==i-1) ? 1 : 0 for i in 1:n, j in 1:n]
end

function plotev(M)
    n = size(M)[1]
    E = eigen(M)
    x = 1:1:n
    display(E.values)
    scatter(x,E.values,legend = false)
end

plotev(pathgraph(30))
```

Eigenvalues and their graphs

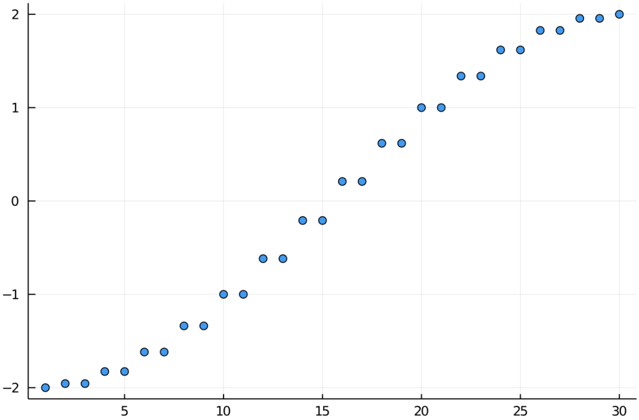


Figure: Who am I?

Eigenvalues and their graphs

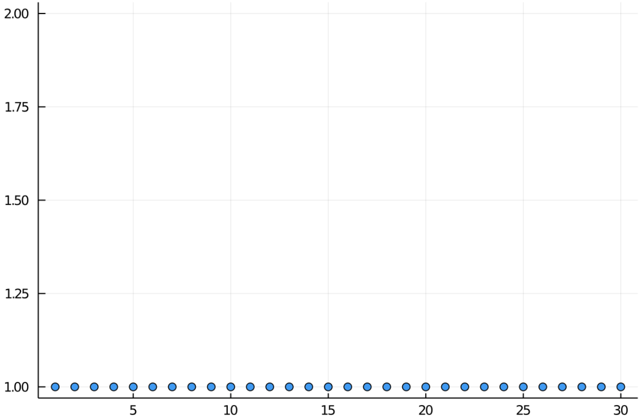


Figure: Who am I?

Eigenvalues and their graphs

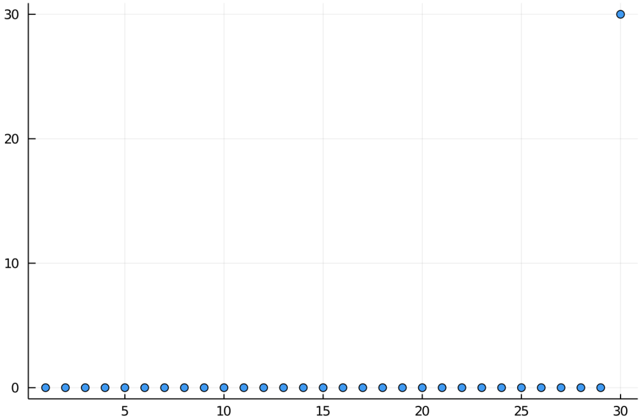


Figure: Who am I?

Eigenvalues and their graphs

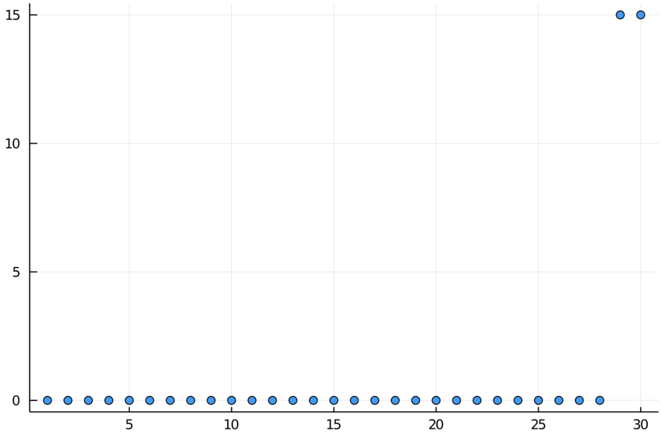


Figure: Who am I?

Eigenvalues and their graphs

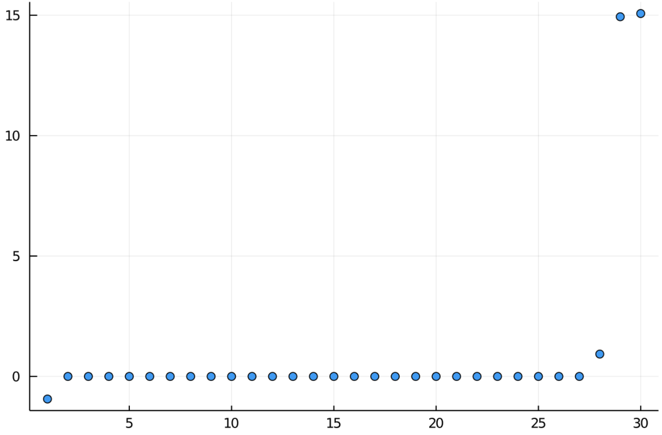


Figure: Who am I? Are these exactly zero?

Eigenvalues and their graphs

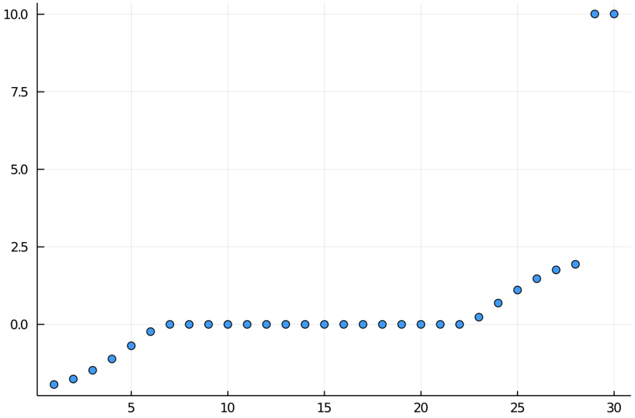


Figure: Eigenvalues of the adjacency matrix **M** of the "third-dumbbell" on 30-vertices

Going back to the clique

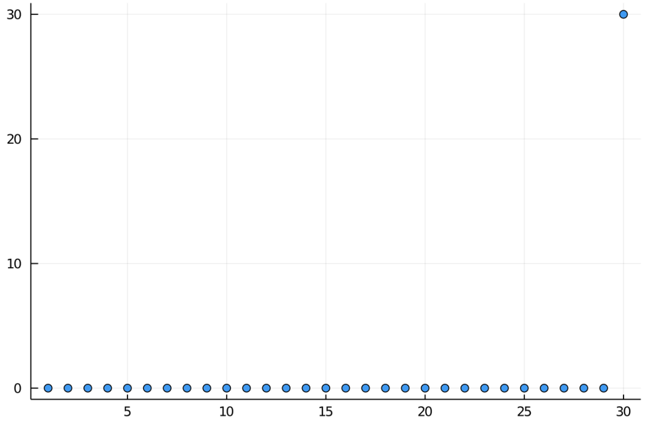


Figure: Clique with self loop, **J**.

Going back to the clique

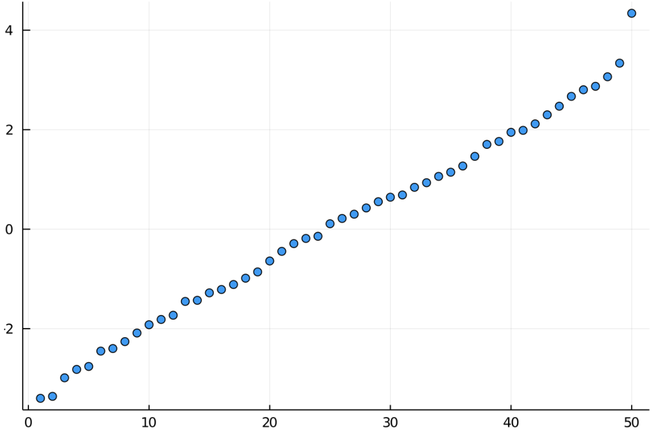
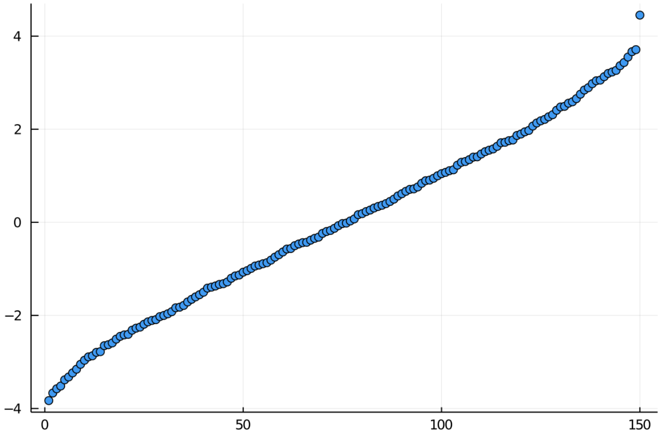


Figure: Every vertex picks 2 neighbors at random; then symmetrize.

Going back to the clique



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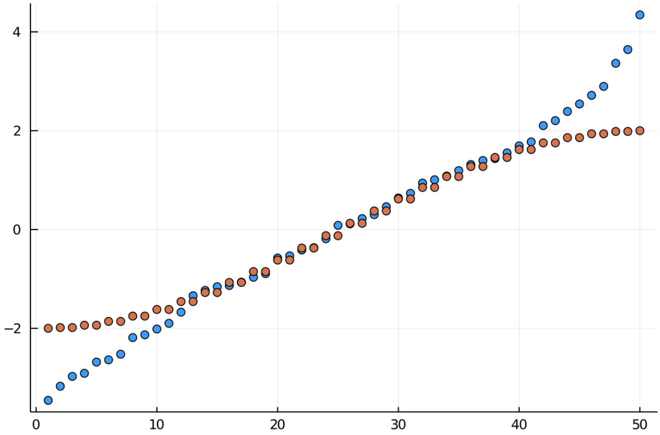

Figure: Cycle vs previous random graph

Interlacing of the Laplacian's eigenvalue

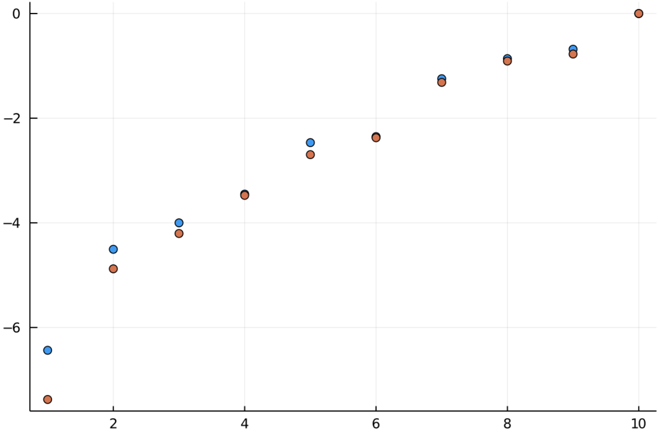

Figure: Interlacing

Last one! Complete binary tree

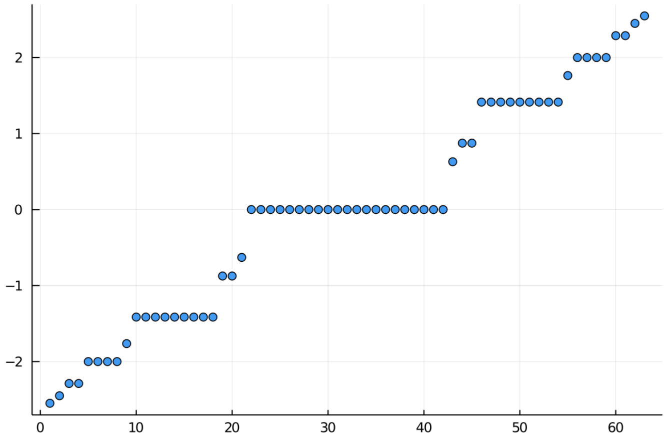

Figure: Depth 5 (in edges) complete binary tree