

# Function Fields

## Unit 9

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# Overview

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# Adjoining algebraic elements

Let  $F/K$  be a field extension, and  $a \in F$  algebraic over  $K$  with minimal polynomial  $f(T) \in K[T]$ . Then,

$$K(a) \cong K[T] / \langle f(T) \rangle.$$

Indeed, consider the ring homomorphism

$$\begin{aligned} \varphi : K[T] &\rightarrow K[a] = K(a) \\ T &\mapsto a \end{aligned}$$

which fixes all elements of  $K$ .

Then,  $\ker \varphi$  consists of all polynomials over  $K$  that vanish at  $a$ . This ideal is generated by  $f(T)$ . The assertion then follows by the first isomorphism theorem.

# Finite extensions are algebraic extensions

Let  $F/K$  be a field extension. Recall that

$$F/K \text{ is finite} \implies F/K \text{ is algebraic.}$$

Indeed, if  $[F : K] = n$  then  $\forall b \in F$ , we have that  $1, b, b^2, \dots, b^n$  are linearly dependent over  $K$ . The (nontrivial) linear relation

$$a_0 + a_1 b + \dots + a_n b^n = 0$$

gives rise to a (nonzero) polynomial over  $K$  with  $b$  as a root.

The converse does not hold in general.

# Algebraic extensions that are finitely generated are finite

Another “finiteness condition” is saying that  $F$  is **finitely generated** over  $K$ , namely,  $F = K(a_1, \dots, a_n)$  for some  $a_1, \dots, a_n \in F$ .

If  $F/K$  is a finite extension then  $F$  is finitely generated over  $K$ .

So, an algebraic extension is a weaker property than finite extension, and same holds for the finitely generated property. However, both together implies finiteness.

## Claim 1

If  $F/K$  is algebraic and finitely generated then it is finite.

# Algebraic extensions that are finitely generated are finite

We sketch the proof of Claim 1.

Consider first a **simple** extension, namely,  $F = K(b)$  for some  $b \in F$ .

As  $F/K$  is algebraic,  $b$  is algebraic, and so if its minimal polynomial is of degree  $d$  then  $1, b, b^2, \dots, b^d$  span  $F$  over  $K$ . Thus,  $[F : K] \leq d$ . In fact,  $[F : K] = d$  as  $1, b, b^2, \dots, b^{d-1}$  are linearly independent over  $K$ .

Consider now the case in which  $F = K(b_1, b_2)$  is an algebraic extension.

Then,  $K(b_1)/K$  is simple and algebraic and so it is finite. Moreover,

$$K(b_1, b_2)/K(b_1) \cong K(b_1)(b_2)/K(b_1)$$

is also algebraic and simple and so it is finite. Thus,

$$[K(b_1, b_2) : K] = [K(b_1, b_2) : K(b_1)] \cdot [K(b_1) : K] < \infty.$$

The general case follows by induction.

# Transcendence degree

## Definition 2 (Algebraic independence)

Let  $F/K$  be a field extension. A set  $T \subseteq F$  is said to be **algebraically independent** over  $K$  if for all distinct  $t_1, \dots, t_n \in T$  and every  $f \in K[T_1, \dots, T_n] \setminus \{0\}$  it holds that

$$f(T_1, \dots, T_n) \neq 0.$$

An algebraically independent set  $T$  is called a **maximal algebraically independent set** if for every  $S \supsetneq T$ ,  $S$  is not algebraically independent.

## Lemma 3

$T$  is a maximal algebraically independent set  $\iff F/K(T)$  is algebraic.

## Definition 4

Let  $F/K$  be a field extension. A maximal algebraically independent set  $T \subseteq F$  is called a **transcendence basis** of  $F$  over  $K$ .

You proved in the recitation that every two transcendence bases have the same cardinality, and so the following definition is sensible.

## Definition 5

Let  $F/K$  be a field extension and let  $T \subseteq F$  be a maximal algebraically independent set of  $F/K$ . The size of  $T$  is called the **transcendence degree** of  $F/K$  and is denoted by  $\text{tr.deg}_K F$ .



# Transcendence degree

Recall our running example

$$f(x, y) = y^2 - x^3 + x \in K[x, y].$$

We defined the ring

$$C_f = K[x, y] / \langle f(x, y) \rangle,$$

and its fraction field

$$K_f = \text{Frac } C_f \cong K(x)[y] / \langle f(x, y) \rangle.$$

**Exercise.** What is  $\text{tr.deg}_K K_f$ ? Give a transcendence basis for  $K_f/K$ .

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## Definition 6 (Field of constants)

Let  $F/K$  be a field extension. The field

$$K' = \{\alpha \in F \mid \alpha \text{ is algebraic over } K\}$$

is called the **field of constants** of  $F/K$ .

Note that  $K \subseteq K' \subseteq F$ .

**Exercise.** Prove that the field of constants of  $K(x)/K$  is  $K$ .

# Algebraic function fields

We turn to define the most basic object in the course.

## Definition 7 (Algebraic function fields)

A field extension  $F/K$  is an **algebraic function field** if

- 1  $F/K$  is finitely generated.
- 2 The field of constants  $K'$  of  $F/K$  is equal to  $K$ .
- 3  $F \neq K$ .

# Algebraic function fields

By (1),  $\text{tr.deg}_K F < \infty$ . Indeed,

$$(1) \implies F = K(a_1, \dots, a_n).$$

Thus, for every  $x \in F$  we can find  $f, g \in K[T_1, \dots, T_n]$  s.t.

$$x = \frac{f(a_1, \dots, a_n)}{g(a_1, \dots, a_n)}.$$

As  $g \neq 0$ , the algebraic relation

$$g(a_1, \dots, a_n)x - f(a_1, \dots, a_n) = 0$$

involves  $x$ . Thus,  $\text{tr.deg}_K F \leq n$ .

# Algebraic function fields

If  $\text{tr.deg}_K F = r$  we say that  $F$  is an **algebraic function field in  $r$  variables** over  $K$ .

**Exercise.** Prove that  $K(x)/K$  is an algebraic function field in one variable over  $K$ .

In this course we focus on algebraic function fields in one variable. For these we have an alternative, seemingly stronger, characterization.

## Claim 8

A field extension  $F/K$  is an algebraic function field in one variable iff the following holds:

- 1  $\exists x \in F$  s.t.  $[F : K(x)] < \infty$ .
- 2 The field of constants  $K'$  of  $F/K$  is equal to  $K$ .
- 3  $F \neq K$ .

## Proof.

Since a finite extension is always finitely generated, condition (1) above implies that  $F$  is finitely generated over  $K(x)$ . Since  $K(x)$  is finitely generated over  $K$ , condition (1) of the definition follows.

# Algebraic function fields

## Proof.

For the other direction, by condition (3) of the definition, there is  $x \in F \setminus K$ . By (2),  $x$  is transcendental over  $K$ .

Since  $F/K$  is an algebraic function field in one variable,  $\text{tr.deg}_K F = 1$  and so  $x$  constitutes a transcendence basis of  $F/K$ . Lemma 3 then implies that  $F/K(x)$  is algebraic.

Now,  $F/K$  is finitely generated by condition (1), and therefore so is  $F/K(x)$ . Claim 1 then implies that  $F/K(x)$  is finite. □



# Function fields

From this point on, we abbreviate and say a **function field** instead of an “algebraic function field in one variable”.

# Why condition (2)?

Condition (1) essentially says we are dealing with one-dimensional objects. Condition (3) avoid trivialities.

If  $K$  is algebraically closed then Condition (2) is vacuously true.

Consider the rational function field  $\mathbb{F}_2(x)$ . If it wasn't for Condition (2) then  $F/\mathbb{F}_2$  where

$$F = \mathbb{F}_2(x)[y] / \langle y^2 + y + 1 \rangle$$

would have been a function field. Indeed,

- It is a field as  $y^2 + y + 1$  is irreducible over  $\mathbb{F}_2(x)$ .
- $F$  is generated by  $y$  over  $\mathbb{F}_2(x)$ .
- $F \neq \mathbb{F}_2$ .

However, convince yourself that

$$F = \mathbb{F}_2(x)[y] / \langle y^2 + y + 1 \rangle \cong \left( \mathbb{F}_2[y] / \langle y^2 + y + 1 \rangle \right) (x) \cong \mathbb{F}_4(x).$$

Thus, we only added new “constants” to  $\mathbb{F}_2(x)$ .

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# Function fields and curves

If  $F/K$  is a function field, by Claim 8,  $\exists x \in F$  s.t.  $[F : K(x)] < \infty$ . Thus,  $F/K(x)$  is algebraic.

Take  $y \in F \setminus K(x)$  if such exists. Let  $\varphi(T) \in K(x)[T]$  be its minimal polynomial over  $K(x)$ . Then,

$$K(x, y) \cong K(x)[T] / \langle \varphi(T) \rangle.$$

If  $y$  happens to satisfy  $F = K(x, y)$  then we get that

$$F \cong K(x)[T] / \langle \varphi(T) \rangle.$$

This is a converse to the way we constructed our example:

$$K_f = \text{Frac } C_f \cong K(x)[y] / \langle y^2 - x^3 + x \rangle.$$

As a side remark, in characteristic 0 we can always find  $y$  as above, and more generally, whenever  $F/K(x)$  is a finite separable extension.

# Function fields and curves

Observe that

$$K(x, y) = K\left(\frac{y}{x}, x\right)$$

and since  $y^2 = x^3 - x$  we get

$$\left(\frac{y}{x}\right)^2 = x - \frac{1}{x}.$$

Thus, if we denote  $z = \frac{y}{x}$  then  $K(x, y) = K(x, z)$  and

$$z^2 = x - \frac{1}{x}.$$

Equivalently,

$$xz^2 = x^2 - 1.$$

So, two different curves may share the same function field.

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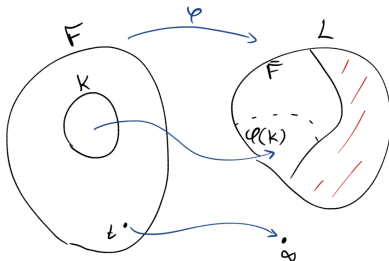
# Places of function fields

## Definition 9

Let  $F/K$  be a function field. A **place of  $F/K$**  is a nontrivial place  $\varphi : F \rightarrow L \cup \{\infty\}$  that is trivial on  $K$ .

## Claim 10

Let  $F/K$  be a function field and  $\varphi : F \rightarrow L \cup \{\infty\}$  a place of  $F/K$  with residue field  $\bar{F} = \varphi(F) \setminus \{\infty\}$ . Then,  $[\bar{F} : \varphi(K)] < \infty$ .



# Places of function fields

Proof.

As  $\varphi$  is nontrivial on  $F$ , we can pick  $t \in F$  with  $\varphi(t) = \infty$ .

Since  $\varphi$  is trivial on  $K$ , we have that  $t \notin K$ . Thus, by Condition (2),  $t$  is transcendental over  $K$ .

Consider the subfield  $K(t)$  of  $F$ , and note that it is isomorphic to the field of rational functions over  $K$ . By Claim 8,  $[F : K(t)] < \infty$ .

$$\begin{array}{c} F \\ | \\ K(t) \\ | \\ K \end{array} \quad \begin{array}{l} \varphi: F \rightarrow L \cup \{\infty\} \\ \quad \quad \quad \quad \quad F \subseteq \\ \\ \varphi|_{K(t)}: K(t) \rightarrow L \cup \{\infty\} \\ \quad \quad \quad \quad \quad K(t) \subseteq \\ \\ \varphi|_K: K \rightarrow L \\ \quad \quad \quad \quad \quad \varphi(K) \subseteq \end{array}$$



# Places of function fields

Proof.

In the recitations you will characterize the places of the rational function field  $K(t)$  and prove that all such places have residue field  $K$ .

The restriction  $\varphi|_{K(t)}$  is such a place. Thus,  $\overline{K(t)} \cong K \cong \varphi(K)$ , and so

$$[\bar{F} : \varphi(K)] = [\bar{F} : \overline{K(t)}].$$

In the recitations you will prove that for every field extension  $E/L$ , and a place  $\psi$  of  $E$ ,

$$[\bar{E} : \bar{L}] \cdot (v_\psi(E^\times) : v_\psi(L^\times)) \leq [E : L].$$

Taking  $E = F$  and  $L = K(t)$ , we conclude that

$$[\bar{F} : \overline{K(t)}] \leq [F : K(t)]$$

which recall is finite.

# Places of function fields

## Definition 11

Let  $F/K$  be a function field. Let  $\varphi : F \rightarrow L \cup \{\infty\}$  be a place of  $F/K$ . Then,  $[\bar{F} : K]$  is called the **degree** of  $\varphi$ , and is denoted by  $\deg \varphi$  (note that we identify  $\varphi(K)$  with  $K$ .)

## Claim 12

If  $\varphi, \varphi'$  are equivalent places of  $F/K$  then  $\deg \varphi = \deg \varphi'$ .

## Proof.

We saw that the residue field of a valuation  $\varphi$  is given by

$$\bar{F}_\varphi = \mathcal{O}_\varphi / \mathfrak{m}_\varphi,$$

and we proved that for  $\varphi, \varphi'$  equivalent it holds that

$$\mathcal{O}_\varphi = \mathcal{O}_{\varphi'} \quad (\text{and so } \mathfrak{m}_\varphi = \mathfrak{m}_{\varphi'}).$$

# Places of function fields

## Definition 13

Let  $F/K$  be a function field. A valuation  $v$  on  $F$  that corresponds to a place  $\varphi$  of  $F/K$  is said to be a **valuation of  $F/K$** .

Note that a valuation on  $F$  is a valuation of  $F/K$  iff  $v(K^\times) = 0$  and  $v(x) < 0$  for some  $x \in F$ .

## Definition 14

A place of a function field  $F/K$  is **discrete** if its corresponding valuation is discrete.

## Claim 15

All places of a function field are discrete.

# Places of function fields

To prove Claim 15, recall a result we proved:

## Claim 16

Let  $\Delta \leq \Gamma$  be ordered groups. Assume that  $(\Gamma : \Delta) < \infty$ . Then,

$$\Delta \cong \mathbb{Z} \implies \Gamma \cong \mathbb{Z}.$$

## Proof.

Let  $\varphi$  be a place of  $F/K$ . Take  $t \in F \setminus K$ . Then,  $t$  is transcendental over  $K$  and  $[F : K(t)] < \infty$ . By the theorem that you will see in the recitations,

$$(v_\varphi(F^\times) : v_\varphi(K(t)^\times)) \leq [F : K(t)] < \infty.$$

You will further prove that all valuations of  $K(t)$  are discrete. Thus, by Claim 16,  $\varphi$  is discrete.

# Places of function fields

Here is another result you will prove in the recitations.

## Claim 17

Let  $F/K$  be a field extension with  $x \in F$  transcendental over  $K$ . Then, there are valuations  $v, v'$  on  $F$  that are trivial on  $K$  s.t.  $v(x) > 0$  and  $v'(x) < 0$ .

In terms of function fields, we immediately get the following result.

## Claim 18

Let  $F/K$  be a function field and let  $x \in F \setminus K$ . Then, there are valuations  $v, v'$  of  $F/K$  with  $v(x) > 0$  and  $v'(x) < 0$ .

This should be read as saying that every non-constant function has a zero and a pole.