AG codes - Spring 2022
Problem Set 01

Gil Cohen Shir Peleg

February 2022

Problem 1. *Let C be a two dimensional Reed Muller code over $\mathbb{F}_q$ as defined in class. Show that C satisfies $\rho \geq \frac{1}{2} - \delta$.*

Problem 2. *Let Γ be an ordered group. Prove that,*

(a) *Let $\gamma \in \Gamma$. $\gamma \geq 0 \Rightarrow -\gamma \leq 0$.*

(b) *Denote $\Gamma_+ = \{\gamma \in \Gamma \mid \gamma \geq 0\}$. Γ_+ is a submonoid.*

(c) $\Gamma_+ \cap (-\Gamma_+) = \{0\}$, $\Gamma_+ \cup (-\Gamma_+) = \Gamma$.

Problem 3. *Let Γ be an abelian group with a submonoid Γ_+ satisfying $\Gamma_+ \cap (-\Gamma_+) = \{0\}$, $\Gamma_+ \cup (-\Gamma_+) = \Gamma$. Define an order on Γ by $\alpha \leq \beta \iff \beta - \alpha \in \Gamma_+$.*

Prove that under this order, Γ is an ordered group.

Problem 4. *Let $I, J \triangleleft R$. Are the following sets ideals? If not, give a counter example and define the smallest ideal that contain it.*

(a) R^*.

(b) $I \cup J$.

(c) $I \cap J$.

(d) $I + J := \{a + b \mid a \in I, b \in J\}$.

(e) $I * J := \{ab \mid a \in I, b \in J\}$.

(f) *If I, J are prime, which of the previous is a prime ideal?*

(g) *If I, J are maximal, which of the previous is a maximal ideal?*

(h) *If I, J are principal, which of the previous is a principal ideal?*

Problem 5. *Let R and S be rings, and $f : R \to S$ be a homomorphism.*

(a) *State which of the following is true, and which is false. No proof needed.*

   (i) *Let $I \triangleleft R$ then $f(I)$ is an ideal in S.*
   (ii) *Let $J \triangleleft S$ then $f^{-1}(J)$ is an ideal in R.*
   (iii) *Let $I \triangleleft R$ be a prime ideal, then $f(I)$ is a prime ideal in S.*
   (iv) *Let $J \triangleleft S$ be a prime ideal, then $f^{-1}(J)$ is a prime ideal in R.*
   (v) *Let $I \triangleleft R$ be a maximal ideal then $f(I)$ is a maximal ideal in S.*
   (vi) *Let $J \triangleleft S$ be a maximal ideal then $f^{-1}(J)$ is a maximal ideal in R.*
   (vii) *Let $I \triangleleft R$ be a principal ideal then $f(I)$ is a principal ideal in S.*
   (viii) *Let $J \triangleleft S$ be a principal ideal then $f^{-1}(J)$ is a principal ideal in R.*

(b) *Characterize the ideals in R/I, i.e, describe the ideals in R/I using only ideals in R.*

Problem 6. *Let R be a ring, and let $a \in R$ be an element.*

Definition.

- *We say that a is a* unit *if there is $a' \in R$ s.t. $a \cdot a' = 1_R$.*
- *We say that a is* irreducible *if it is not the product of two non units.*
- *We say that a is* prime *if for every $c, d \in R$, if $a | c \cdot b$ then $a | c$ or $a | b$.*
- *We say that R is a* Unique factorial domain (UFD) *if every non-zero element a of R can be written as a product (an empty product if a is a unit) of irreducible elements p_i of R and a unit u: $x = u \cdot p_1 \cdot p_2 \cdots p_n$ with $n \geq 0$ and this representation is unique in the following sense: If $q_1, \ldots, q_m$ are irreducible elements of R and w is a unit such that*

  $x = w \cdot q_1 \cdot q_2 \cdots q_m$ with $m \geq 0$ then $m = n$, and there exists a bijective map $\varphi : [n] \to [m]$ such that for every $i \in [n]$, $p_i = u_i q_{\varphi(i)}$ for some irreducible element $u_i \in R$.

(a) *Prove that a is prime $\iff$ (a) is a prime ideal.*

(b) *Prove that in a domain every prime element is irreducible. Show that the converse holds in a UFD.*

Problem 7. *Let S be a set of polynomials with coefficient in $\mathbb{Z}$ that have no linear term (that is, that the coefficient of x is 0).*

(a) *Prove that $S \subset \mathbb{Z}[x]$ is a subdomain.*

(b) *Prove that $x^6 \in S$ can be written as a product of irreducibles in two different ways, deduce that S is not a UFD.*

(c) *Find a polynomial that is irreducible in S but is not prime in S.*