Recitation 4: The Places of the Rational Function Field

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1 Function Fields

Definition 1. A field extension F/K is called an *algebraic function field in one variable over* K (or simply a *function field*) if the following holds:

1. There exists $x \in F$ such that $[F : K(x)] < \infty$.
2. $K \subsetneq F$.
3. The field of constants of F/K is equal to K.

Example 2. Let K be a field and let t be transcendental over K. Then

$$K(t) := \left\{ \frac{f(t)}{g(t)} \,\middle|\, f, g \in K[t],\ g \neq 0 \right\}$$

is called the field of rational functions in variable t over K. Then $K(t)/K$ is a function field, and is called the rational function field over K.

Remark 1. For $u = \frac{f(t)}{g(t)} \in K(t)$ we define $\deg u := \deg f - \deg g$ (check that it is well-defined).

Definition 3. Let F, L be fields. A *place* is a map $\varphi \colon F \to L \cup \{\infty\}$ satisfying:

1. $\varphi(1) = 1$.
2. $\varphi(a + b) = \varphi(a) + \varphi(b)$, provided that $\{\varphi(a), \varphi(b)\} \neq \{\infty\}$.
3. $\varphi(ab) = \varphi(a)\varphi(b)$, provided that $\{\varphi(a), \varphi(b)\} \neq \{0, \infty\}$.

A place is called *trivial* if $\varphi(a) \neq \infty$ for all $a \in F$, i.e. $\varphi(F) \subseteq L$.

Definition 4. Two places $\varphi_1 \colon F \to L_1 \cup \{\infty\}$ and $\varphi_2 \colon F \to L_2 \cup \{\infty\}$ are *equivalent* if for all $a \in F$,

$$\varphi_1(a) \neq \infty \iff \varphi_2(a) \neq \infty.$$

Definition 5. A *place* of a function field F/K is a non-trivial place $\varphi \colon F \to L \cup \{\infty\}$ that is trivial on K. The *residue field* of φ is the field $\overline{F} := \varphi(F) \backslash \{\infty\}$.

Remark 2. Note that since φ is trivial on K we have $K \cong \varphi(K) \subseteq L$. Identifying $\varphi(K)$ with K via φ, we may assume that $K \subseteq L$ and that φ is the identity on K.

2 The Places of $K(t)/K$

What are the places $\varphi\colon K(t) \to L \cup \{\infty\}$ of the rational function field $K(t)/K$?

In class we have already seen one example of such places:

Example 6. Let $p(t) \in K[t]$ be a monic irreducible polynomial. Then $L := K[t]/\langle p(t)\rangle$ is a field. Let

$$\begin{aligned} \pi\colon K[t] &\to K[t]/\langle p(t)\rangle \\ g(t) &\longmapsto \overline{g(t)} := g(t) + \langle p(t)\rangle \end{aligned}$$

be the natural projection. Notice that

$$g(\bar{t}) = 0 \iff \overline{g(t)} = 0 \iff g(t) \in \langle p(t)\rangle \iff p(t) \mid g(t) \text{ in } K[t].$$

In particular, $\bar{t}$ is a root of $p(t)$ in L, and we can extend π to a place

$$\varphi_p\colon K(t) \to L \cup \{\infty\}$$

as follows: Given $u(t) \in K(t)$, write $u(t) = \frac{f(t)}{g(t)}$ with $f(t), g(t) \in K[t]$ coprime and define

$$\varphi_p(u(t)) := \begin{cases} \frac{f(\bar{t})}{g(\bar{t})} & p(t) \nmid g(t) \text{ in } K[t] \\ \infty & \text{otherwise} \end{cases}.$$

That is, the place φ_p "substitutes" the root $\bar{t} \in L$ of $p(t)$ in $u(t)$.

Exercise 1. Show that if $p(t), q(t) \in K[t]$ are two distinct, monic irreducible polynomials in $K[t]$, then the places φ_p and φ_q are not equivalent.

Theorem 7. *Let $\varphi\colon K(t)/K \to L\cup\{\infty\}$ be a place of $K(t)/K$ such that $\varphi(t) \neq \infty$. Then φ is equivalent to φ_p for some monic irreducible polynomial $p(t) \in K[t]$. Thus, there is a one-to-one correspondence between the monic irreducible polynomials in $K[t]$ and the (equivalent classes of) places φ of $K(t)/K$ in which $\varphi(t) \neq \infty$.*

Proof. Let $\varphi\colon K(t)/K \to L \cup \{\infty\}$ be a place of $K(t)/K$ such that $\varphi(t) \neq \infty$. By Remark 2, we may assume that $K \subseteq L$ and that φ is the identity on K. Since $\varphi(t) \neq \infty$ we have $\varphi(K[t]) \subseteq L$ and the restriction $\varphi_0 := \varphi|_{K[t]}\colon K[t] \to L$ is a ring homomorphism. Since L is a field, $\ker(\varphi_0)$ is a prime ideal of $K[t]$. Thus, either $\ker(\varphi_0) = \{0\}$ or $\ker(\varphi_0) = \langle p\rangle$ for some unique monic, irreducible polynomial $p \in K[t]$.

Case 1. $\ker(\varphi_0) = \{0\}$.

Then for every $\frac{f}{g} \in K(t)$ we have $\varphi(g) \notin \{0, \infty\}$, hence

$$\varphi\left(\frac{f}{g}\right) \cdot \varphi(g) = \underbrace{\varphi(f)}_{\in L} \implies \varphi\left(\frac{f}{g}\right) = \frac{\varphi_0(f)}{\varphi_0(g)} \in L.$$

However, this implies that φ is trivial on $K(t)$, which is forbidden.

Case 2. $\ker(\varphi_0) = \langle p \rangle$ for some monic, irreducible $p \in K[t]$. Let $\tau := \varphi(t)$. Then $\tau \in L$ is a root of p, as

$$\varphi(p) = p(\varphi(t)) = p(\tau) = 0$$

and if $h \in K[t]$ is such that $p \nmid h$ in $K[t]$, then

$$\varphi(h) = h(\varphi(t)) = h(\tau) \in L^\times.$$

Now, every $u \in K(t)^\times$ has a unique representation

$$u = p^m \cdot \frac{f}{g} \tag{1}$$

with $f, g \in K[t] \backslash \{0\}$ coprime, $p \nmid f$, $p \nmid g$ (in $K[t]$) and $m \in \mathbb{Z}$. It follows that

$$\varphi(u) = \begin{cases} \frac{f(\tau)}{g(\tau)} & m = 0 \\ 0 & m > 0 \\ \infty & m < 0 \end{cases} \tag{2}$$

The corresponding valuation ring is

$$\mathcal{O}_p = \{u \in K(t) \mid \varphi(u) \neq \infty\} = \left\{ \frac{f}{g} \,\middle|\, f, g \in K[t],\, p \nmid g \right\}$$

so φ is equivalent to the place φ_p. □

Remark 3. The maximal ideal of $\mathcal{O}_p$ is

$$\mathfrak{m}_p = \{u \in K(t) \mid \varphi(u) = 0\} = \left\{ \frac{f}{g} \,\middle|\, f, g \in K[t],\, p \nmid g,\, p \mid f \right\}$$

and a corresponding valuation is the *p-adic valuation*: $\nu_p(u) = m$ (where u admits the representation (1)).

Exercise 2. Show that the residue field $\varphi(\mathcal{O}_p)$ of the place φ defined in (2) is isomorphic to $K[t]/\langle p \rangle$. In particular, it is a finite extension of K of degree $\deg p$.

It remains to find the places $\varphi\colon K(t)/K \to L \cup \{\infty\}$ in which $\varphi(t) = \infty$.

In this case, we must have $\varphi(t^{-1}) = 0$. Each $u \in K(t)^\times$ can be written as

$$u = \frac{\sum_{i=0}^{k} a_i t^i}{\sum_{j=0}^{\ell} b_j t^j}$$

with $a_k, b_\ell \neq 0$. Then

$$u = t^{k-\ell} \cdot \frac{\sum_{i=0}^{k} a_i t^{i-k}}{\sum_{j=0}^{\ell} b_j t^{j-\ell}}.$$

Note that φ maps the quotient in the RHS to $a_k/b_\ell \in K^\times$. Hence

$$\varphi(u) = \begin{cases} a_k/b_\ell & k = \ell \\ 0 & k < \ell \\ \infty & k > \ell \end{cases}$$

This indeed gives a place, which we denote by φ_∞. The corresponding valuation ring is

$$\begin{aligned} \mathcal{O}_\infty &= \{u \in K(t) \mid \varphi_\infty(u) \neq \infty\} \\ &= \left\{ \frac{f}{g} \,\middle|\, f, g \in K[t],\ \deg f \leq \deg g \right\} \\ &= \left\{ u \in K(t) \,\middle|\, f, g \in K[t],\ \deg u \leq 0 \right\} \end{aligned}$$

and a corresponding valuation is $\nu_\infty(u) = -\deg(u)$. It is also clear that the residue field $\varphi_\infty(K(t)) \backslash \{\infty\}$ is equal to K, and that φ_∞ is not equivalent to any φ_p where $p(t) \in K[t]$ is a monic irreducible polynomial.