Affine Curves

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Definition

Let K be any field. An affine curve over K is a pair $(\mathrm{Max}(A), A)$ where

- A is a f.g. K-algebra, and
- $\dim(A) = 1$.

If A is a domain, we say that the curve $(\mathrm{Max}(A), A)$ is irreducible.

Discussion

We think of

- $\mathrm{Max}(A)$ *as the points of the curve, and*
- *of A as the functions that are defined on the entire curve.*
- *The evaluation of a function $\alpha \in A$ at a point $P \in \mathrm{Max}(A)$, denoted by $\alpha(P)$, is defined to be the element $\alpha + P \in A/P$.*

Discussion

The "geometric" example that motivated this definition is what we have been calling the affine curve $Z_f(\bar{K})$ where $f \in \bar{K}[x, y]$.

Consider the affine curve $(\mathrm{Max}(C_f), C_f)$. By Hilbert's Nullstellensatz,

$$Z_f(\bar{K}) \rightleftarrows \mathrm{Max}(C_f).$$

So we can indeed interpret $\mathrm{Max}(C_f)$ as points and C_f as functions defined at all points in this case. Indeed, we proved that:

- *C_f is a f.g. $\bar{K}$-algebra, and*
- $\dim(C_f) = 1$.
- *Moreover, when f is irreducible, C_f is a domain and so the curve $(\mathrm{Max}(C_f), C_f)$ is irreducible.*

Discussion

- $\dim(A) = 1$ *is a requirement from any affine curve* $(\text{Max}(A), A)$.
- *By Hilbert's basis theorem, A is a noetherian ring since it is a f.g. K-algebra.*
- *In general, A does not have to be integral (the one thing missing for A to be a Dedekind domain).*
- *We proved that C_f is a Dedekind domain $\iff$ $Z_f(\bar{K})$ is nonsingular. This motivates the following definition.*

Definition

A nonsingular affine curve is an affine curve $(\max(A), A)$ such that A is a Dedekind domain.