Graphs and their matrices

Gil Cohen

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Overview

Graphs and their matrices

Unless otherwise stated, our graphs will always be finite and undirected, sometimes simple, sometimes weighted.

We will use matrices in two different ways:

- as an operator $\mathbf{x} \mapsto \mathbf{M}\mathbf{x}$; and
- defining a quadratic form $\mathbf{x} \mapsto \mathbf{x}^T\mathbf{M}\mathbf{x} = \sum_{i,j} \mathbf{M}_{i,j}\mathbf{x}_i\mathbf{x}_j$.

The adjacency matrix

Perhaps the most natural matrix associated with a graph G is its adjacency matrix $\mathbf{M}_G$ that is given by

$$\mathbf{M}_G(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

We index the rows and columns of $\mathbf{M}_G$ by V.

The path graph

0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0
0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	1	0

The diffusion operator

The most natural operator associated with a graph G is probably its diffusion operator.

Let $\mathbf{D}_G$ be the diagonal matrix with $\mathbf{D}_G(v, v) = \deg v$. For weighted graphs, $\deg v$ is the sum of weights over edges incident to v.

Assuming there are no isolated vertices, $\mathbf{D}_G$ is invertible and we define the diffusion operator by

$$\mathbf{W}_G = \mathbf{M}_G \mathbf{D}_G^{-1}.$$

When G is d-regular then $\mathbf{D}_G = d\mathcal{I}$ and so $\mathbf{W}_G = \frac{1}{d}\mathbf{M}_G$.

The diffusion operator

Think of a vector $\mathbf{p} \in \mathbb{R}$ specifying how much stuff there is at each vertex. After one time step, the "distribution" of stuff is $\mathbf{W}_G \mathbf{p}$.

One important example is when $\mathbf{p}$ is a probability distribution. The operator $\mathbf{W}_G$ then captures taking a random step.

Sometimes we will be interested in a *lazy random walk*

$$\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathbf{W}_G.$$

Overview

The Laplacian

The Laplacian $\mathbf{L}_G = \mathbf{D}_G - \mathbf{M}_G$ yields the most natural quadratic form associated with a graph.

$$\mathbf{x}^T \mathbf{L}_G \mathbf{x} = \sum_{uv \in E} (\mathbf{x}(u) - \mathbf{x}(v))^2.$$

This measures the "smoothness" of $\mathbf{x}$ across the edges of G.

To see this, note that the Laplacian of the graph connecting only u, v is given by

$$\mathbf{L}_{uv} = (\delta(u) - \delta(v))(\delta(u) - \delta(v))^T$$

$$\implies \mathbf{x}^T \mathbf{L}_{uv} \mathbf{x} = \mathbf{x}^T (\delta(u) - \delta(v))(\delta(u) - \delta(v))^T \mathbf{x} = (\mathbf{x}(u) - \mathbf{x}(v))^2.$$

Then use linearity: $\mathbf{L}_G = \sum_{uv \in E} \mathbf{L}_{uv}$.

Overview

Eigenvalues and eigenvectors

Recall that a nonzero vector ψ is an eigenvector of a matrix $\mathbf{M}$ with eigenvalue λ if

$$\mathbf{M}\psi = \lambda\psi.$$

Hence λ is an eigenvalue of $\mathbf{M}$ if

- $\lambda\mathcal{I} - \mathbf{M}$ is singular;
- λ is a root of the characteristic polynomial of $\mathbf{M}$, $\det(x\mathcal{I} - \mathbf{M})$.

Eigenvalues and eigenvectors

Let G be a graph. We think of a vector as a map $\psi : V \to \mathbb{R}$.

ψ is an eigenvector of $\mathbf{M}_G$ with eigenvalue λ iff for every $v \in V$

$$\sum_{uv \in E} \psi(u) = \lambda \cdot \psi(v).$$

Every real (or even complex) matrix **M** has n eigenvalues in $\mathbb{C}$, counted with multiplicities. Sometimes, though, we will be short of eigenvectors. Eigenvalues and eigenvectors are suitable for studying symmetric matrices (or matrices similar to symmetric matrices). Otherwise, one should consider singular values and singular vectors.

Theorem (The Spectral Theorem)

Let **M** be an $n \times n$ real, symmetric matrix. Then there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (not necessarily distinct) and n mutually orthogonal unit vectors $\psi_1, \dots, \psi_n$ such that ψ_i is an eigenvector of **M** of eigenvalue λ_i.

Overview

Eigenvalues and their graphs

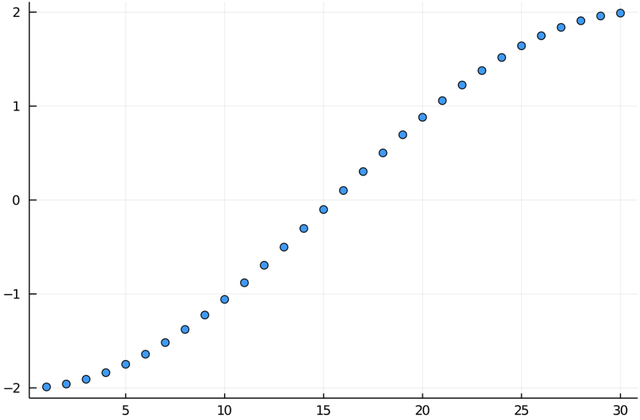

Figure: Eigenvalues of the adjacency matrix **M** of the 30-vertex path graph (largest is $\lambda_{30} \sim 1.989$).

Eigenvalues and their graphs

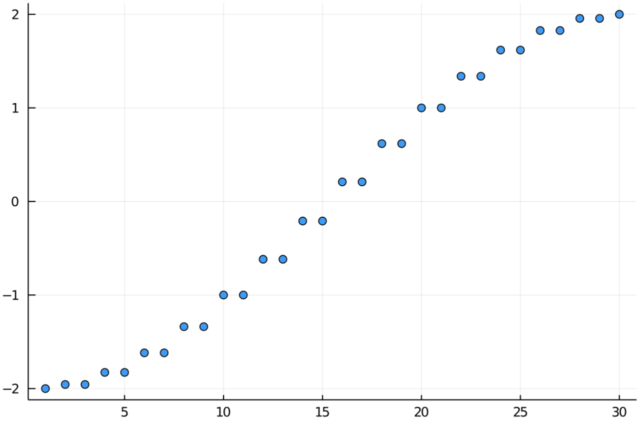

Figure: Eigenvalues of the adjacency matrix **M** of the 30-vertex cycle.

Eigenvalues and their graphs

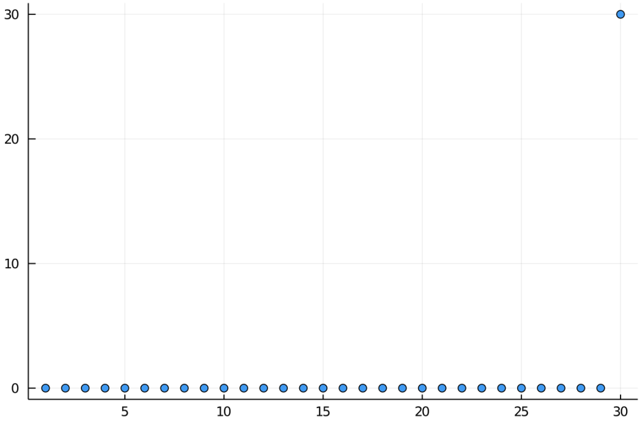

Figure: Who am I?

Eigenvalues and their graphs

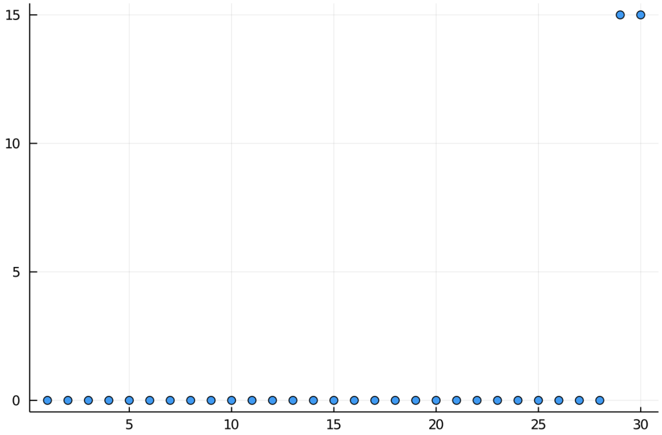

Figure: Who am I?

Eigenvalues and their graphs

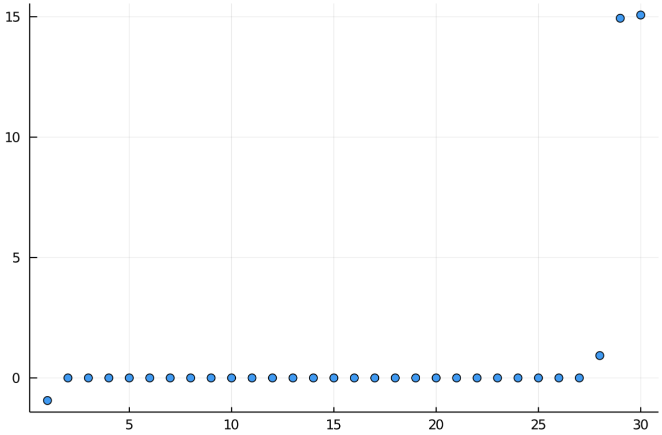

Figure: Adding a single edge

Complete binary tree

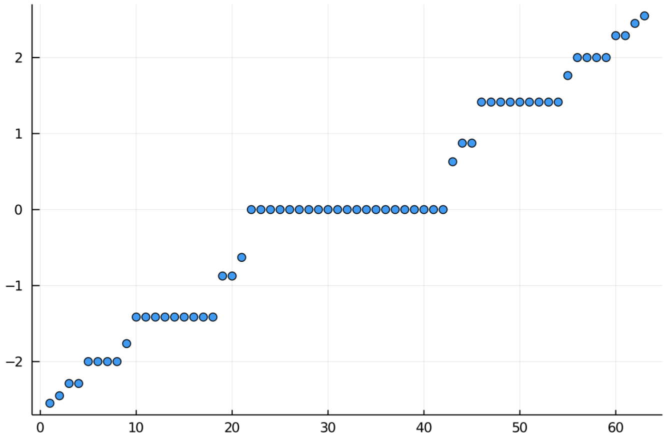

Figure: Depth 5 (in edges) complete binary tree

Interlacing of the Laplacian's eigenvalue

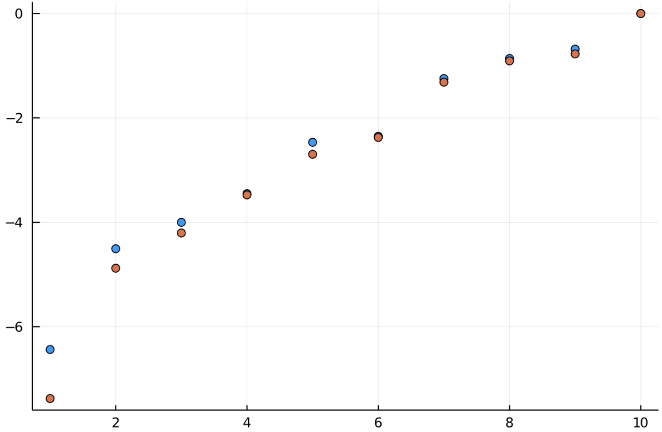

Figure: Interlacing

Summary

Notation

For a graph G we denote by

- $\mathbf{M}_G$ its adjacency matrix.
- $\mathbf{W}_G = \mathbf{M}_G \mathbf{D}_G^{-1}$ its random walk matrix.
- $\mathbf{L}_G = \mathbf{D}_G - \mathbf{M}_G$ its Laplacian.

The eigenvalues are denoted by μ_i, ω_i and λ_i, respectively.

For d-regular graphs

- $\mathbf{W}_G = \frac{1}{d}\mathbf{M}_G$ and $\omega_i = \frac{\mu_i}{d}$.
- $\mathbf{L}_G = d\mathcal{I} - \mathbf{M}_G$ and $\lambda_i = d - \mu_i$.

Summary