Recitation 5: Teleportation

In the lecture we saw the protocol enabling Alice to teleport one qubit of information using one EPR pair shared between her and Bob and two classical bits she is sending to him. We saw it as a concatenation of two protocols: one to perform a CNOT and another that teleports assuming the CNOT has been performed.

1. (a) Draw the overall circuit for the teleportation protocol. In order to draw a classically-controlled gate (meaning a gate that is acting only if a measurement's outcome is a specific outcome, usually $|1\rangle$), use a meter and a double-wire (see Figure 5.1).

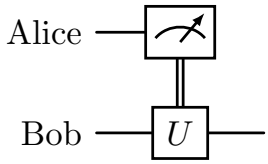

Figure 5.1: Circuit for measuring the first register and applying U on the second register only if the measurement's outcome was a specific outcome (usually $|1\rangle$).

Answer. The circuit is drawn in Figure 5.2. △

(b) Note that Bob's role is only to apply or not apply two gates to his part of the shared EPR pair, according to the classical bits he gets from Alice. On the other hand, Alice is performing two measurements and reports to Bob their outcomes.

Alice measures both her qubits after some unitary transformations, or said differently, she performs a complete measurement of her 4-dimensional quantum space in a different basis than the computational one. What is the basis in which she measures? If we think of them as two separate partial measurements, what are they? How do they correspond to the gates Bob has to apply, depending on their outcomes?

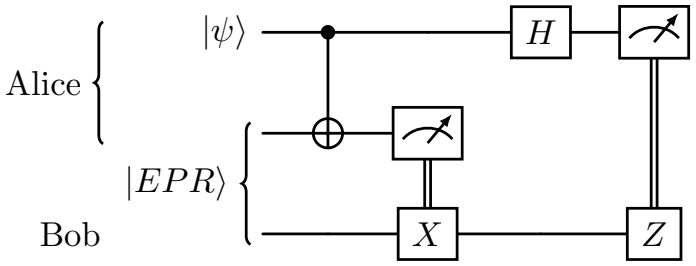

Figure 5.2: The overall circuit for teleportation

Answer. We saw in the previous section that measuring in the computational basis after performing a unitary U, when we are only interested in the "classical outcome" (as we are here), is equivalent to measuring in the basis $\{U^\dagger |i\rangle\}_{i=0}^{d-1}$. So all we need to do is apply the inverse of the unitary Alice applies to her qubits before she measures. She acts with $H_1 \circ CNOT_{1,2}$. We know both the gates are Hermitian, on top of being Unitary, meaning they are their own inverses. So the inverse is $CNOT_{1,2} \circ H_1$. Lets apply this to each of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and by that we'll have calculated the basis in which Alice measures.

$$\begin{aligned} |v_{00}\rangle &= CNOT_{1,2} \circ H_1 |00\rangle \\ &= CNOT_{1,2} |+0\rangle \\ &= CNOT_{1,2} \tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\ &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \end{aligned}$$

and similarly

$$|v_{01}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |v_{10}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle), |v_{11}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

These are called the Bell states and are sometimes denoted by $|\Phi^+\rangle = |v_{00}\rangle, |\Phi^-\rangle = |v_{10}\rangle, |\Psi^+\rangle = |v_{01}\rangle, |\Psi^-\rangle = |v_{11}\rangle$ (note that $|\Phi^+\rangle = |EPR\rangle$ and $|\Psi^-\rangle$ is the singlet state from question 2 in the Teleportation part of Ex3).

If we use the same logic, we can understand each of the partial measurements as well. But we can also use the fact that we know the basis in which together they are a complete measurement, as each one will have to split the 4 states into two equal parts (because we know each one gives one split of the space and together they split the 4 dimensions into 1 dimensions). We can see that performing $CNOT_{1,2}$ on two of them makes the second qubit a certain $|0\rangle$ and performing it on the other two makes it a certain $|1\rangle$. The first measurement is

thus a $|\Phi^{\pm}\rangle$ vs $|\Psi^{\pm}\rangle$ measurement. Then we see that performing H_1 after the $CNOT$ takes the first qubit to either a certain $|0\rangle$, for the $|\bullet^{+}\rangle$ states, and to a certain $|1\rangle$ for the $|\bullet^{-}\rangle$ states. This makes sense with the correction Bob needs to apply to his qubit – the first measurement tells whether a bit-flip happened and the second – whether a phase-flip happened (as we already saw in the lecture). △

(c) Look at the projections of the initial state $|\psi\rangle_1 |EPR\rangle_{2,3}$ [1] onto the basis you found in the previous subquestion and use this to reprove the correctness of the teleportation protocol.

Answer. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be the qubit state to be teleported.
$\langle\Phi^{+}|_{1,2}|\psi\rangle_1|\Phi^{+}\rangle_{2,3} = \frac{1}{2}(\langle 00|_{1,2} + \langle 11|_{1,2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) = \frac{1}{2}(\alpha|0\rangle_3 + \beta|1\rangle_3) = \frac{1}{2}|\psi\rangle_3$ so indeed in this case Bob needs to perform no corrections.
$\langle\Phi^{-}|_{1,2}|\psi\rangle_1|\Phi^{+}\rangle_{2,3} = \frac{1}{2}(\langle 00|_{1,2} - \langle 11|_{1,2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) = \frac{1}{2}(\alpha|0\rangle_3 - \beta|1\rangle_3) = \frac{1}{2}Z_3|\psi\rangle_3$ so we see the phase correction that Bob needs to perform.
$\langle\Psi^{+}|_{1,2}|\psi\rangle_1|\Phi^{+}\rangle_{2,3} = \frac{1}{2}(\langle 01|_{1,2} + \langle 10|_{1,2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) = \frac{1}{2}(\alpha|1\rangle_3 + \beta|0\rangle_3) = \frac{1}{2}X_3|\psi\rangle_3$ so we see the bit correction Bob needs to perform.
$\langle\Psi^{-}|_{1,2}|\psi\rangle_1|\Phi^{+}\rangle_{2,3} = \frac{1}{2}(\langle 01|_{1,2} - \langle 10|_{1,2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) = \frac{1}{2}(\alpha|1\rangle_3 - \beta|0\rangle_3) = \frac{1}{2}X_3Z_3|\psi\rangle_3$ so we see that Bob needs to perform a bit correction and then a phase correction. △

2. (b) Alice holds $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$. Show how she can teleport $|\psi\rangle$ to Bob using 4 classical bits of communication, assuming they share two EPR pairs. What is the generalization to n qubits?

Answer. Alice can use the teleportation protocol for each qubit separately. This is a general claim regarding the teleportation protocol T, that it works not only for isolated qubit states, but also for qubits that are entangled with $n-1$ other ones: $(\alpha|\psi_0\rangle_{A_{1\ldots n-1}}|0\rangle_{A_n} + \beta|\psi_1\rangle_{A_{1\ldots n-1}}|1\rangle_{A_n})|\Phi^{+}\rangle_{A_{n+1}B} \mapsto \alpha|\psi_0\rangle_{A_{1\ldots n-1}}|\varphi\rangle_{A_nA_{n+1}}|0\rangle_B + \beta|\psi_1\rangle_{A_{1\ldots n-1}}|\varphi\rangle_{A_nA_{n+1}}|1\rangle_B$ for $|\varphi\rangle$ the Bell state outcomes of Alice's measurement. Measuring the A_nA_{n+1} qubits in the Bell basis the quantum outcomes are (before normalisation):

$$\underbrace{\tfrac{1}{\sqrt{2}}(|0x\rangle_{A_{n,n+1}} + (-1)^z|1\overline{x}\rangle_{A_{n,n+1}})}_{|\varphi\rangle}\tfrac{1}{\sqrt{2}}(\langle 0x|_{A_{n,n+1}} + (-1)^z\langle 1\overline{x}|_{A_{n,n+1}})$$

$$(\alpha|\psi_0\rangle_{A_{1\ldots n-1}}|0\rangle_{A_n} + \beta|\psi_1\rangle_{A_{1\ldots n-1}}|1\rangle_{A_n})\tfrac{1}{\sqrt{2}}(|00\rangle_{A_{n+1}B} + |11\rangle_{A_{n+1}B}) =$$
$$= \tfrac{1}{2\sqrt{2}}(\alpha|\psi_0\rangle_{A_{1\ldots n-1}}|\varphi\rangle_{A_{n,n+1}}|x\rangle_B +$$
$$+ (-1)^z\beta|\psi_1\rangle_{A_{1\ldots n-1}}|\varphi\rangle_{A_{n,n+1}}|\overline{x}\rangle_B)$$

[1] the subscripts help to track the different registers and prevent misunderstanding without the use of the symbol $\otimes$

and by the teleportation protocol, if $x = 1$ Bob performs a bit-flip, and if $z = 1$ he then performs a phase-flip so the final state is as claimed and the teleported qubit state is intact as part of the initial big state, but now on Bob's qubit. $\triangle$