Valuations

Unit 4

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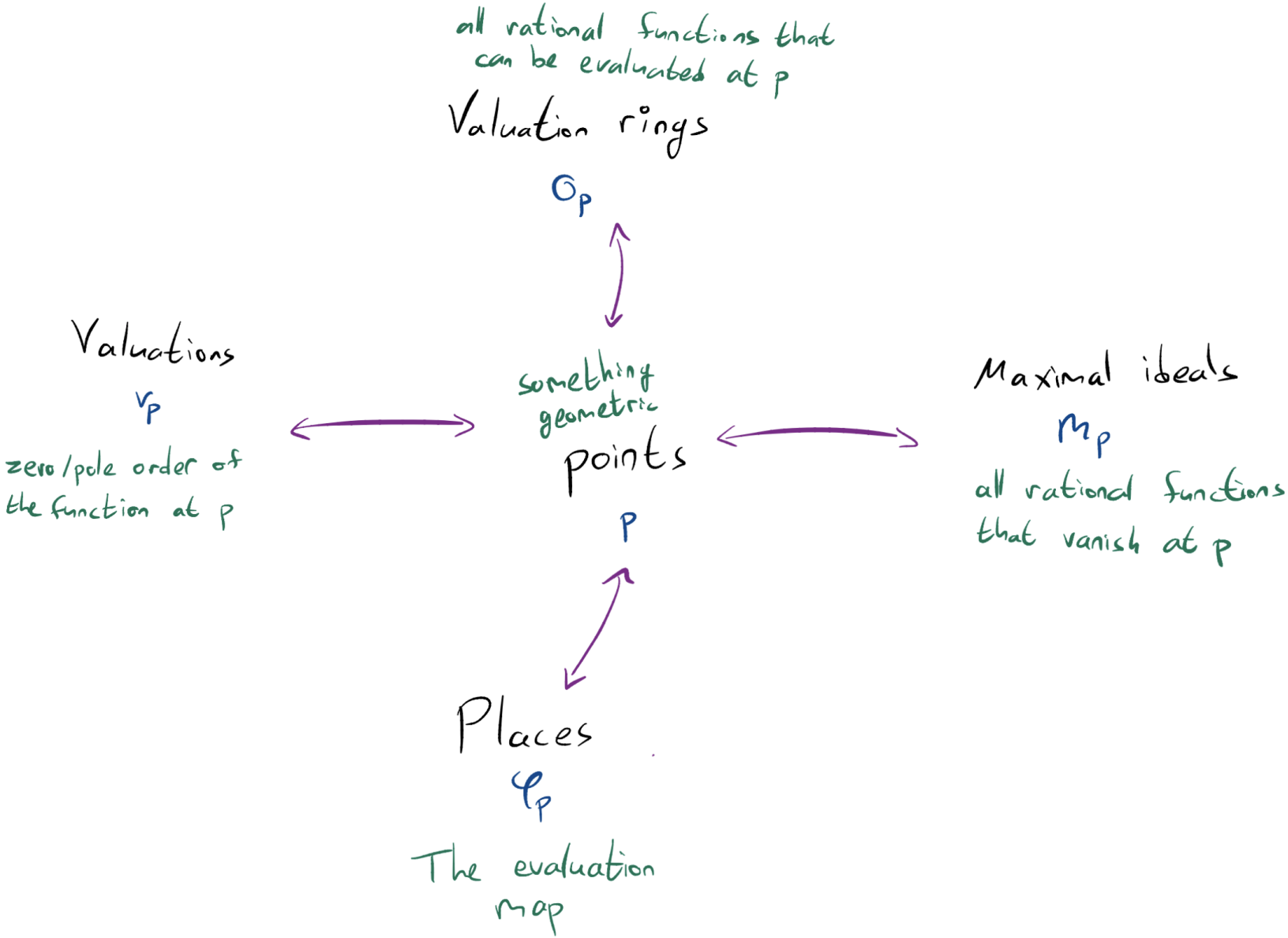
February 27, 2022

Recall


all rational functions that can be evaluated at p
Valuation rings
$\mathcal{O}_p$
Valuations
v_p
zero/pole order of the function at p
something geometric
points
p
Maximal ideals
$\mathfrak{m}_p$
all rational functions that vanish at p
Places
φ_p
The evaluation map

Overview

Valuations

Definition 1 (Valuations)

A valuation on a field F is map $v : \mathsf{F}^\times \to \Gamma$, where Γ is an ordered group, satisfying

1. additivity: $v(ab) = v(a) + v(b)$. That is, v is a group homomorphism from the multiplicative group of F to Γ.
2. triangle inequality: $v(a+b) \geq \min(v(a), v(b))$,

assuming $a, b, a+b \in \mathsf{F}^\times$.

It is convenient to extend the definition to $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ by setting

$$v(a) = \infty \quad \iff \quad a = 0.$$

Doing so, we do not have to assume anything about $a, b, a+b$ as $\infty > \Gamma$.

Overview

p-adic valuations of $\mathbb{Q}$

A trivial example for a valuation is given by the trivial valuation $v : F^\times \to \{0\}$. Let's look at more interesting examples.

Let $p \in \mathbb{N}$ be a prime. We define

$$v_p : \mathbb{Q}^\times \to \mathbb{Z}$$

as follows. Given $q \in \mathbb{Q}^\times$, write

$$q = p^n \cdot \frac{a}{b}$$

with $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime to p. As $\mathbb{Z}$ is UFD, n is unique. We define the p-adic valuation by

$$v_p(q) = n.$$

Exercise. What is $v_3\left(\frac{5}{9}\right)$?

$p(x)$-adic valuations of $\mathrm{K}(x)$

Let K be a field and $p(x) \in \mathrm{K}[x]$ irreducible. We define

$$\upsilon_{p(x)} : \mathrm{K}(x)^\times \to \mathbb{Z}$$

as follows. Let $q(x) \in \mathrm{K}(x)^\times$. Write

$$q(x) = p(x)^n \cdot \frac{a(x)}{b(x)}$$

with $a(x), b(x) \in \mathrm{K}[x] \setminus \{0\}$ and $a(x), b(x)$ coprime to $p(x)$.

By the UFD-ness of $\mathrm{K}[x]$, n is unique. We define the $p(x)$-adic valuation by

$$\upsilon_{p(x)}(q(x)) = \upsilon_p(q) = n.$$

If $p(x)$ is of the form $p(x) = x - \alpha$ for some $\alpha \in K$, and $q(x) \in \mathrm{K}(x)^\times$ then $\upsilon_p(q) = \upsilon_{x-\alpha}(q)$ is the multiplicity of α as a root or pole of q. We sometimes write υ_α for short.

Overview

Valuations and absolute values

Let $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ a valuation and assume $\Gamma \lhd \mathbb{R}$. For $a \in \mathsf{F}^\times$ define

$$|a| = 2^{-v(a)}.$$

Informally, "$2^{-v(0)} = 2^{-\infty} = 0$", and so we extend $|\cdot|$ to F by setting $|0| = 0$. We have that

$$|ab| = 2^{-v(ab)} = 2^{-(v(a)+v(b))} = 2^{-v(a)}2^{-v(b)} = |a||b|.$$

Further, by the triangle inequality,

$$|a+b| = 2^{-v(a+b)} \leq 2^{-\min(v(a),v(b))} = \max(2^{-v(a)}, 2^{-v(b)}) = \max(|a|, |b|),$$

and so

$$|a + b| \leq \max(|a|, |b|) \leq |a| + |b|.$$

Valuations and absolute values

Therefore, a valuation gives rise to an absolute value. In fact, it yields a stronger notion called a non-Archimedean absolute value: not only is

$$|a+b| \leq |a| + |b|$$

but also

$$|a+b| \leq \max(|a|, |b|).$$

Otherwise, an absolute value is called Archimedean.

We think of a valuation as measuring closeness. It is instructive and productive to study the topology induced by the valuation but in this course we will not proceed in this direction.

Several results establish that "close" polynomials have similar properties. E.g., if $f(x)$ is separable and irreducible then so is any sufficiently close $g(x)$.

Overview

Basic properties of valuations

Claim 2

Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation. Then,

1. $v(1) = 0$
2. $v(a) = v(-a)$. In particular, $v(-1) = 0$.
3. $v(a^{-1}) = -v(a)$.
4. If $v(a) \neq v(b)$ then
$$v(a + b) = \min(v(a), v(b)).$$
5. $v(\sum_i a_i) \geq \min(v(a_i))$.
6. If $\sum_{i=1}^{n} a_i = 0$ and $n > 1$ then $\exists i \neq j$ with $v(a_i) = v(a_j)$.

Basic properties of valuations

The proof of the claim is straightforward and is left as an exercise. We only prove (4)–the strict triangle inequality.

Proof.

Assume $v(a) < v(b)$. We know that $v(a+b) \geq v(a)$ by the triangle inequality but we wish to prove that $v(a+b) = v(a)$. Now,

$$\begin{aligned} v(a) &= v(a+b-b) \\ &\geq \min\left(v(a+b), v(-b)\right) \\ &= \min\left(v(a+b), v(b)\right). \end{aligned}$$

Hence, if $v(a+b) > v(a)$ we would get $v(a) > v(a)$. □

Basic properties of valuations

Claim 3

Let R be a domain and Γ an ordered group. Assume $v : \mathrm{R} \to \Gamma \cup \{\infty\}$ satisfies

1. $v(ab) = v(a) + v(b)$,
2. $v(a+b) \geq \min(v(a), v(b))$,
3. $v(a) = \infty \iff a = 0$

for all $a, b \in \mathrm{R}$. Then v can be extended to a valuation in a unique way to Frac R.

Proof sketch.

For $a \in \mathrm{R}$, the value of $v(a^{-1})$ is determined as $v(a^{-1}) = -v(a)$. Thus, the only possible extension of v would map $\frac{a}{b}$ to $v(a) - v(b)$. It is left to show that this is indeed a valuation.

Overview

Discrete valuations

Observe that if $\upsilon : \mathsf{F}^\times \to \Gamma$ is a valuation then $\upsilon(\mathsf{F}^\times)$ is an ordered subgroup of Γ. Thus, sometimes one replaces Γ with $\upsilon(\mathsf{F}^\times)$. This allows us to assume essentially wlog that υ is onto.

Definition 4 (Discrete valuations)

When $\upsilon(\mathsf{F}^\times) \cong \mathbb{Z}$ we say that υ is a discrete valuation.

By the above remark, we sometimes "normalize" a nontrivial discrete valuation so that $\upsilon(\mathsf{F}^\times) = \mathbb{Z}$.

Overview

Some facts from Ring Theory

We recall the following basic definitions from ring theory.

Definition 5 (Units)

Let R be a domain. An element $a \in \mathrm{R}$ is called a unit if a has an inverse in R, namely,

$$\exists b \in \mathrm{R} \quad ab = ba = 1.$$

The set of units of R is denoted by $\mathrm{R}^\times$. Note that $\mathrm{R}^\times$ is a group under multiplication and is referred to as the unit group of R.

Definition 6 (Irreducible elements)

Let R be a domain. A non-zero non-unit element $a \in \mathrm{R}$ is irreducible if whenever $a = bc$ then b or c is a unit.

Definition 7 (Prime elements)

Let R be a domain. A non-zero non-unit element $a \in \mathrm{R}$ is prime if whenever $a \mid bc$ then $a \mid b$ or $a \mid c$.

Some facts from Ring Theory

We recall that every prime is irreducible.

One can show (excluding the annoying $a = 0$ case) that

$$\begin{aligned} a \in \mathrm{R} \text{ is prime} &\iff \langle a \rangle \text{ is a prime ideal} \\ &\iff \mathrm{R}/\langle a \rangle \text{ is a domain.} \end{aligned}$$

Perhaps slightly less known (yet trivial) is the equivalence

$$a \in \mathrm{R} \text{ is irreducible} \iff \langle a \rangle \text{ is maximal amongst principal ideals.}$$

Some facts from Ring Theory

As mentioned, a prime element is always irreducible. The converse holds in UFD. Thus, in a UFD

$$a \in \mathrm{R} \text{ is irreducible} \iff \mathrm{R}/\langle a \rangle \text{ is a domain.}$$

You may have seen that

$$\mathrm{R} \text{ is a UFD} \implies \mathrm{R}[x] \text{ is a UFD.}$$

Let K be a field. $\mathrm{K}[x]$ is a UFD (it is in fact a Euclidean domain $\implies$ PID $\implies$ UFD). Thus, so is $\mathrm{K}[x, y]$.

We can thus conclude that

$$\mathrm{K}[x, y] \Big/ \langle f(x, y) \rangle \text{ is a domain} \iff f(x, y) \text{ is irreducible.}$$

Example

Let K be a field. Let

$$f(x,y) = y^2 - x^3 + x \in K[x,y],$$

and consider the curve

$$Z_f = \{(x,y) \in \mathrm{K} \times \mathrm{K} \mid f(x,y) = 0\}.$$

$f(x,y)$ is irreducible as otherwise $\exists g(x) \in \mathrm{K}[x]$ s.t.

$$f(x,y) = (y - g(x))(y + g(x)).$$

But then $g(x)^2 = x^3 - x$ which is impossible by degree considerations. Thus,

$$C_f = \mathrm{K}[x,y] \Big/ \langle f(x,y) \rangle$$

is a domain. We denote its fraction field by K_f.

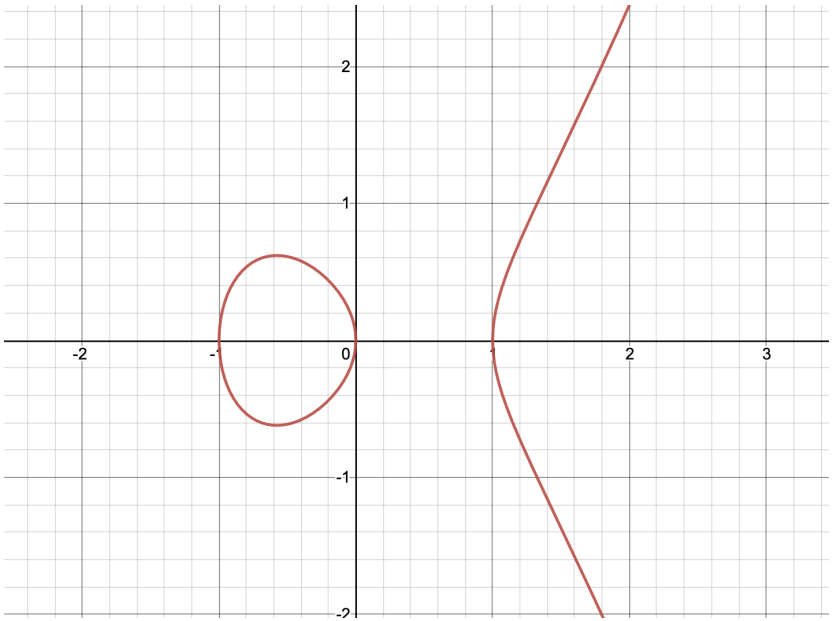
Example

The point $o = (0, 0)$ is on the curve Z_f as $f(0, 0) = 0$. Therefore, we may try to find a valuation $v_o : K_f \to \mathbb{Z} \cup \{\infty\}$.

We abuse notation and write x, y also for the projections of x, y via the map

$$K[x, y] \to C_f = K[x, y] \Big/ \langle y^2 - x^3 + x \rangle.$$

We do not yet have the tools to prove it, but trust me that such v_o exists with $v_o(x) = 2$.



Example

Let's see what we can make of it. As $y^2 = x^3 - x$ in C_f,

$$v_o(y^2) = v_o(x^3 - x).$$

I promised you that $v_o(x) = 2$, and so

$$v_o(x^3) = 3v_o(x) = 6.$$

Thus, by the strict triangle inequality,

$$v_o(x^3 - x) = \min(v_o(x^3), v_o(x)) = 2.$$

Thus,

$$2v_o(y) = v_o(y^2) = 2,$$

and so we conclude that $v_o(y) = 1$.

Example

As $y^2 = x^3 - x$ in C_f, every element of $K_f = \operatorname{Frac} C_f$ can be written in the form

$$\frac{a(x) + b(x)y}{c(x) + d(x)y}$$

with $a(t), b(t), c(t), d(t) \in \mathrm{K}[t]$ and with the understanding that x, y are the images in C_f as discussed above. We can further simplify

$$\frac{a(x) + b(x)y}{c(x) + d(x)y} = \frac{a(x) + b(x)y}{c(x) + d(x)y} \cdot \frac{c(x) - d(x)y}{c(x) - d(x)y} = A(x) + B(x)y,$$

where $A(t), B(t) \in \mathrm{K}(t)$.

To summarize, every element in K_f is of the form

$$A(x) + B(x)y$$

with $A(t), B(t) \in K(t)$.

Example

Since $v_o(y) = 1$,

$$v_o(A(x) + B(x)y) \geq \min\left(v_o(A(x)), 1 + v_o(B(x))\right).$$

Assume now further that K is a finite field, $\mathrm{K} = \mathbb{F}_q$. Then, $v_o(z) = 0$ for every $z \in \mathrm{K}$. Indeed, we have that $z^q = z$ for every $z \in \mathrm{K}$, and so

$$v_o(z) = v_o(z^q) = q v_o(z) \quad \Longrightarrow \quad v_o(z) = 0.$$

As $v_o(x) = 2$, we get by the strict triangle inequality that $v_o(g(x))$ is even for every $g(x) \in \mathrm{K}[x]$. Indeed,

$$v_o(x^i) = i v_o(x) = 2i.$$

Thus, if $g(x) = \sum_{i=0}^{d} a_i x^i$, then

$$v_o(g(x)) = 2j$$

where j is the least integer such that $a_j \neq 0$.

Example

Thus, $v_o(g(x))$ is even for every $g(x) \in K(x)$. Now, recall that

$$v_o(A(x) + B(x)y) \geq \min\left(v_o(A(x)), 1 + v_o(B(x))\right).$$

As $v_o(A(x)), v_o(B(x))$ are even, and since $v_o(y) = 1$ we get, by the strict triangle inequality that

$$v_o(A(x) + B(x)y) = \min\left(v_o(A(x)), 1 + v_o(B(x))\right).$$

Some random examples include

$$v_o\left(\frac{1}{x} + y\right) = \min(-2, 1) = -2,$$

and

$$v_o\left(y^2 - x\right) = v_o(x^3 - 2x) = \min(6, 2) = 2.$$

Example

Consider the function $\frac{x}{y} \in C_f$. Is it defined at $(0,0)$?

Using valuations, we see that

$$v_o\left(\frac{x}{y}\right) = v_o(x) - v_o(y) = 2 - 1 = 1,$$

and so $\frac{x}{y}$ is not only defined but also vanishes at $(0,0)$.

To see it in a different way, recall that $y^2 = x^3 - x$ in C_f and so

$$\frac{x}{y} = \frac{y}{x^2 - 1}.$$

Thus, using this representation it is easier to see that at $(0,0)$ the function $\frac{x}{y}$ evaluates to 0.

Example

Some recommended exercises:

Exercise. Prove that C_f is not a UFD. Hint:

$$y^2 = x^3 - x = x(x-1)(x+1).$$

Exercise. Recall that $\mathsf{K}_f = \operatorname{Frac} C_f$, where

$$C_f = \mathsf{K}[x,y] \Big/ \langle y^2 - x^3 + x \rangle.$$

Prove that

$$\mathsf{K}_f \cong \mathsf{K}(x)[y] \Big/ \langle y^2 - x^3 + x \rangle.$$

Overview

Equivalent valuations

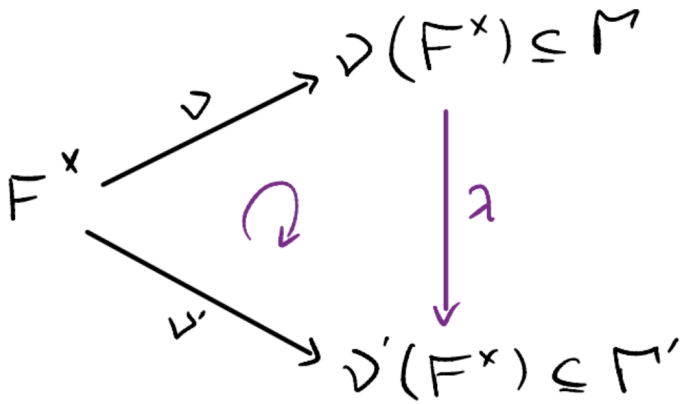
Definition 8

Two valuations $v : F^\times \to \Gamma, v' : F^\times \to \Gamma'$ are equivalent if

$$\forall a \in F^\times \quad v(a) \geq 0_\Gamma \iff v'(a) \geq 0_{\Gamma'}.$$

Claim 9

v, v' are equivalent $\iff$ $\exists$ an order-preserving group isomorphism $\lambda : v(F^\times) \to v'(F^\times)$ s.t. $v' = \lambda \circ v$.



Equivalent valuations

Proof.

For the nontrivial direction, as v, v' are equivalent,

$$v(a) = 0 \quad \iff \quad v'(a) = 0.$$

Indeed, assume $v(a) = 0$. Then, in particular,

$$v(a) \geq 0 \quad \implies \quad v'(a) \geq 0.$$

Moreover, $v(a^{-1}) = -v(a) = 0 \geq 0$ and so $v'(a^{-1}) \geq 0$, implying $v'(a) \leq 0$. Overall then, $v'(a) = 0$.

Therefore, $\ker \nu = \ker \nu'$, and so by the first isomorphism theorem for groups,

$$v(\mathsf{F}^\times) \cong \mathsf{F}^\times \Big/ \ker v = \mathsf{F}^\times \Big/ \ker v' \cong v'(\mathsf{F}^\times).$$

This then induces a group isomorphism $\lambda : v(\mathsf{F}^\times) \to v'(\mathsf{F}^\times)$, and I leave it for you to show it is order preserving. □