Algebraic Geometric Codes

Recitation 03

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The field of rational functions

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We want to study all the valuations of $F(x)$.
Let $F \subseteq K$ be a field, and let $\varphi : F \to K \cup \{\infty\}$ be a place, trivial on F (i.e. for $\alpha \in F$,$\varphi(\alpha) = \alpha$.

Valuations over $F[x]$

Assume $\varphi(x) \neq \infty$. Then, as φ is a place, we deduce that for every $f \in F[x]$, $\varphi(f) \neq \infty$. Thus $\varphi_0 = \varphi\mid_{F[x]}: F[x] \to K$.

Claim

Let L be a field and $\psi : F[x] \to L$ be an homomorphism, then $ker(\psi) = \langle p \rangle$ for some irreducible polynomial p.

Proof.

$\square$

Valuations over $F[x]$ – cont

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If $p = 1$ then $ker(\varphi_0) = F[x]$ which is a contradiction to the assumption that φ is trivial on F.

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If $p = 1$ then $ker(\varphi_0) = F[x]$ which is a contradiction to the assumption that φ is trivial on F.
Thus, $\deg(p) \geq 1$. Denote by $\tau = \varphi(x) \in K$. We can represent each element $u \in F(x)$ in the form $u = p^m \frac{f(t)}{g(t)}$, where p, f, g are pairwise co-prime and $m \in \mathbb{Z}$. Therefore

$$\varphi(u) = \begin{cases} \frac{f(\tau)}{g(\tau)} & m = 0 \\ 0 & m > 0 \\ \infty & m < 0 \end{cases}.$$

Therefore $\mathcal{O}_p = \{f/g \mid p \nmid g\}$.

Valuations over $F[x]$ – cont

Using the previous representation ($u = p^m \frac{f(t)}{g(t)}$) we can define a valuation $v_p(u) = m$. It's valuation ring is also $\mathcal{O}_p$ and thus it corresponds to φ. This is called the p-adic valuation of $F(x)$.

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Claim

If $p \neq q$ are coprime irreducible polynomials, then v_p and v_q are not equivalent.

Proof.

As $v_p(1/p) = -1$ and $v_q(1/p) = 0$.

□

Other Valuations

The only case left to consider is when $\varphi(t) = \infty$ therefore $\varphi(\frac{1}{t}) = 0$. Every $u \in F(x)$ can be written

$$u = \frac{\sum_{i=0}^{k} a_i x^i}{\sum_{j=0}^{l} b_i x^i} = x^{k-l} \frac{\sum_{i=0}^{k} a_i x^{i-k}}{\sum_{j=0}^{l} b_i x^{i-l}}$$

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$\mathcal{O}_\infty = \{u \mid \deg(u) \leq 0\}$, and $v_\infty(u) = -\deg(u)$.

Conclusion - calculating degrees

We concluded that the set of valuations of $F(x)$ is

$$\{v_\infty\} \cup \{v_p \mid p \in F[x] \text{ is irreducible}\}.$$

What are the degrees of the corresponding places? $\deg(P_\infty) = 1$, $\deg(P_p) = \deg(p)$.