Introduction to the Ramanujan Graphs
part of the seminar

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November 30, 2021

Outline

The seminar so far

Recall that a d-regular graph G is called Ramanujan if all eigenvalues of the corresponding adjacency matrix $\mathbf{M}_G$, but for the largest one $\mu_1 = d$, are between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$. That is,

$$\mu = \max(|\mu_2|, |\mu_n|) \leq 2\sqrt{d-1}.$$

When used loosely, saying that G is an expander means that $\mu < \alpha d$ for some constant $\alpha < 1$.

Some known facts on Ramanujan graphs

1. The Alon-Boppana bound: $\mu \geq 2\sqrt{d-1} - o(1)$ (Nilli 1991), presented by Itamar on Week 4.
2. Infinite families of Ramanujan graphs of every degree $d = p + 1$ exist with p a prime. These are strongly explicit super elegant constructions (Margulis, Lubotzky-Phillips-Sarnak 1988). A lecture to define, a year to analyze.
3. A random d-regular graph is very close to Ramanujan, namely, $\mu \leq 2\sqrt{d-1} + \varepsilon$ (Friedman 2008).

Some known facts on Ramanujan graphs

Based on lifting of graphs:

1. Bilu-Linial gave a weakly-explicit construction of every degree d achieving $\mu = O(\sqrt{d} \cdot \log^{3/2} d)$.
2. They proved the existence of graphs with $\mu = O(\sqrt{d} \cdot \sqrt{\log d})$.
3. Bilu-Linial raised a conjecture that, if holds, proves the existence of Ramanujan graphs of every degree.
4. Mohanty, O'Donnell and Paredes 2020 gave a weakly explicit construction acheiving $\mu = 2\sqrt{d-1} + \varepsilon$ of every degree d.

All these results, in lifting-based arguments in general, can handle every degree d but not every graph size, as the size doubles.

Some known facts on Ramanujan graphs

As for strongly explicit constructions

1. The line graph construction (presented by Lin and Cohav on Week 5) gives strongly explicit expander graphs.
2. A more elaborate variant using the so-called Zig-Zag product (Reingold-Vadhan-Wigderson 2002) can be used to give $\mu = O(d^{2/3})$. Alternatively, one can use the derandomized squaring operation by Rozenman and Vadhan 2005.
3. An improved construction by Ben-Aroya and Ta-Shma 2011 gives $\mu = O(\sqrt{d} \cdot 2^{\sqrt{\log d}}) = d^{1/2+o(1)}$.

Each of (2),(3) played a key role in breakthrough works in derandomization (Reingold 2005) and coding theory (Ta-Shma 2017).

Outline

The Bilu-Linial conjecture

Before we begin, for a superb talk on this by Spielman, search "Ramanujan graphs of every degree - Daniel Spielman" on YouTube, or go straight to

`https://www.youtube.com/watch?v=D_KvIVusdIM&t=2s`

The Bilu-Linial conjecture

As we saw on Week 7 by Kedem and Itay's talk, Bilu and Linial considered the 2-lifts of a graph. They (easily) showed that the spectrum of the lifted graph is the union of the spectrum of the original graph and the spectrum of the signed adjacency matrix.

This potentially allows one to double the size of the graph while maintaining both the degree and the bound on μ.

The Bilu-Linial conjecture states that every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$.

The (first) work of Marcus, Spielman and Srivastava

Marcus, Spielman and Srivastava (2015) proved the Bilu-Linial conjecture for bipartite graphs.

First, note that every 2-lift of a bipartite graph is a bipartite graphs, and so this approach of doubling the size “stays within” the family of bipartite graphs.

The thing about bipartite graphs is that their spectrum is symmetric around the origin. Thus, what MSS proved is that for every d-regular graph there exists a 2-lift for which all new eigenvalues are bounded above by $2\sqrt{d-1}$.

MSS proof strategy

Let $p_s(x)$ be the characteristic polynomial of the signed adjacency matrix $(\mathbf{M}_G)_s$ that corresponds to a 2-lift s.

Consider the expected characteristic polynomial

$$p(x) = \mathbb{E}_s[p_s(x)].$$

MSS strategy is as follows:

1. Prove that the max root of $p(x)$ is at most $2\sqrt{d-1}$. Note that it is not even clear that $p(x)$ is real rooted!
2. The polynomials $\{p_s(x)\}_s$ form an interlacing family.
3. (1)+(2) $\implies$ that the max root of $p_s(x)$, for some 2-lift s, is bounded by $2\sqrt{d-1}$.

The matching polynomial

To prove (1), MSS observed that the expected characteristic polynomial

$$p(x) = \mathbb{E}_s[p_s(x)]$$

is in fact a well-studied polynomial called the matching polynomial.

$$p(x) = \sum_i (-1)^i m_i x^{n-2i},$$

where m_i is the number of matchings in G with exactly i edges.

Heilmann-Lieb 1972 proved that all roots of $p(x)$ are real and that the largest root, in absolute value, is bounded above by $2\sqrt{d-1}$.

This will be covered in Week 12 by Guy and Amir (Chapter 45 in Spielman).

MSS proof strategy

Recall the MSS strategy:

1. Max root of $p(x)$ is at most $2\sqrt{d-1}$.
2. $\{p_s\}_s$ form an interlacing family.
3. (1)+(2) $\implies$ max root of p_s, for some 2-lift s, is bounded by $2\sqrt{d-1}$.

As for (2),(3), next week (Week 9) Ronel and Yuval will follow Chapter 42 in Spielman and talk about these points.

Yoav and Dvir will wrap it up in the last meeting (Week 13).

Outline

Another work another talk

Again, for a great talk by Spielman on this work search "Daniel Spielman - Ramanujan Graphs and Free Probability" on YouTube or go to

`https://www.youtube.com/watch?v=JAHFLZ0dX00`

The (second) work of Marcus, Spielman and Srivastava

We now consider a different paper by the same Marcus, Srivastava and Spielman (Interlacing families IV: Bipartite Ramanujan graphs of all sizes) which confusingly enough has a conference version in 2015 which is the same year of the journal publication of the above work by the same authors.

MSS15b proved that for every degree d and every number of vertices n there exists a d-regular bipartite Ramanujan (multi) graph with n vertices.

MSSb proof strategy

In the seminar (as in Spielman's book), we are only going to bound μ_2 by $2\sqrt{d-1}$ which is sometimes called "half Ramanujan".

Consider the sum of d random matchings on n vertices. Note that the characteristic polynomial of a perfect matching is

$$(x-1)^{n/2}(x+1)^{n/2}.$$

regardless of the perfect matching itself.

MSSb proof strategy

Let $\mathbf{M}_1, \ldots, \mathbf{M}_d$ be the adjacency matrix of d random, independent (perfect) matchings on n vertices. Let

$$\mathbf{M} = \mathbf{M}_1 + \cdots + \mathbf{M}_d.$$

Consider the characteristic polynomial $p(x) = \mathbb{E} p_{\mathbf{M}}(x)$.

The MSSb startegy is as follows:

1. Prove that $p(x)$ is real rooted.
2. Prove that the second largest root of $p(x) \leq 2\sqrt{d-1}$.
3. Prove, again using interlacing, that there exist d perfect matchings for which the corresponding characteristic polynomial has second largest eigenvalue that is bounded by that of $p(x)$.

MSSb proof strategy

1. Prove that p is real rooted.
2. Prove that the second largest root of $p \leq 2\sqrt{d-1}$.
3. Prove, using interlacing, that there exist d perfect matchings for which the corresponding characteristic polynomial has second largest eigenvalue that is bounded by that of p.

(1) will be proven next week (Week 9) by Ronel and Yuval following Chapter 42.

In Week 10, will do some ground work (which we discuss shortly next). Using it, in Week 11 Maya and Tommy will do step (2), and deduce (3).

What will I present in Week 10?

Consider two symmetric matrices $\mathbf{A}, \mathbf{B}$ such that $\mathbf{A}\mathbf{1} = a\mathbf{1}$ and $\mathbf{B}\mathbf{1} = b\mathbf{1}$. Let the corresponding characteristic polynomials be $(x-a)p(x)$ and $(x-b)q(x)$, respectively.

MSSb established a formula for the characteristic polynomial of $\mathbf{A} + \mathbf{\Pi}\mathbf{B}\mathbf{\Pi}^{\mathsf{T}}$ where $\mathbf{\Pi}$ is the matrix that corresponds to a random permutation on n elements.

In Week 10 I will analyze this following Chapter 43. In fact, MSSb developed an analog of a fairly modern branch of mathematics called free probability theory for their result, dubbed finite free probability.

See Adam Marcus talk (`https://www.youtube.com/watch?v=iCkWndYh32M`) if you want to learn more about this.

Outline

The semicircle law

Consider an $n \times n$ symmetric random matrix $\mathbf{A}_n$ such that each entry is ± 1 with equal probability, independently across entries (up to the symmetry condition).

It is a fundamental result in Random Matrix Theory that the (normalized) eigenvalue distribution of the matrix obeys, in the limit $n \to \infty$, the semicircle law.

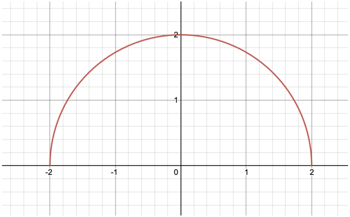

Figure: What is wrong with this picture?

The semicircle law

Consider the moments of the distribution of the eigenvalues which are given by

$$m_k(n) = \mathbb{E}[\mathrm{Tr}(\mathbf{A}_n^k)].$$

By symmetry, $m_k(n) = 0$ for odd k. It is a well-known, fundamental result in Random Matrix Theory that

$$m_{2k} = \lim_{n \to \infty} m_{2k}(n) = C_k,$$

where C_k is the k^{th} Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$.

The Cauchy transform

Given a distribution μ on $\mathbb{R}$ we define the Cauchy transform G_μ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

for all $z \in \mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

It is a well-known fact that G_μ is an analytic function that maps $\mathbb{C}^+$ to $\mathbb{C}^+$.

One reason that the Cauchy transform is so useful is that it encodes the moments of μ when considered around ∞ and the density of μ when approaching the real line.

The Cauchy transform

One can recover the measure μ from G_μ by approaching the real line:

$$-\frac{1}{\pi}\lim_{\varepsilon\searrow 0}\int_a^b \mathrm{Im}(G_\mu(x+i\varepsilon))dx = \mu((a,b)) + \frac{1}{2}\mu(\{a,b\}).$$

On the other hand, if μ is compactly supported, namely, $\mu([-r,r]) = 1$ for some $r > 0$ then G_μ has a power series expansion (around ∞) as given by

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}$$

for all $z \in \mathbb{C}^+$ with $|z| > r$.

Outline

Free probability theory

Free probability abstracts in a very powerful way key aspects of limit behavior in Random Matrix Theory (and in other fields which I know very little about).

Consider the example of a random $n \times n$ matrix $\mathbf{A}_n$ from before. Let a be a formal variable and define $\varphi : \mathbb{C}[a] \to \mathbb{C}$ to be the unital $\mathbb{C}$-linear map that encodes the limit behavior of the moments

$$\varphi(a^k) = \lim_{n\to\infty} \mathbb{E}[\mathrm{Tr}(\mathbf{A}_n^k)].$$

Free probability theory

Given another random $n \times n$ matrix $\mathbf{B}_n$ such that the $\mathbf{A}$ matrices are independent of the $\mathbf{B}$ matrices, we extend φ to $\varphi : \mathbb{C}[a, b] \to \mathbb{C}$ such that

$$\varphi(a^k) = \lim_{n\to\infty} \mathbb{E}[\mathrm{Tr}(\mathbf{A}_n^k)],$$
$$\varphi(b^k) = \lim_{n\to\infty} \mathbb{E}[\mathrm{Tr}(\mathbf{B}_n^k)].$$

Moreover, φ encodes the mixed moments via the corresponding monomial. E.g.,

$$\varphi(aba) = \lim_{n\to\infty} \mathbb{E}[\mathrm{Tr}(\mathbf{A}_n\mathbf{B}_n\mathbf{A}_n)],$$
$$\varphi(ab^2a^3b^4) = \lim_{n\to\infty} \mathbb{E}[\mathrm{Tr}(\mathbf{A}_n\mathbf{B}_n^2\mathbf{A}_n^3\mathbf{B}_n^4)].$$

Free probability theory

Even when $(\mathbf{A}_n)_n$ is independent of $(\mathbf{B}_n)_n$ there is not reason that the marginals $(\varphi(a^k))_k, (\varphi(b^k))_k$ will determine the mixed moments such as $\varphi(abab)$.

The key idea underlying free probability theory is that in many settings there is an additional property, dubbed freeness, that enables one to deduce the mixed moments from the marginals.

We say that a, b are free if for every monomial

$$\varphi((a^{i_1} - \varphi(a^{i_1})1)(b^{j_1} - \varphi(b^{j_1})1) \cdots (a^{i_k} - \varphi(a^{i_k})1)(b^{j_k} - \varphi(b^{j_k}))1) = 0.$$

Freeness - examples

We say that a, b are free if for every monomial

$$\varphi((a^{i_1}-\varphi(a^{i_1})1)(b^{j_1}-\varphi(b^{j_1})1)\cdots(a^{i_k}-\varphi(a^{i_k})1)(b^{j_k}-\varphi(b^{j_k}))1)=0.$$

For example, if a, b are free then

$$\varphi((a-\varphi(a)1)(b-\varphi(b)1))=0.$$

However, by linearity,

$$\begin{aligned}\varphi((a-\varphi(a)1)(b-\varphi(b)1)) &= \varphi(ab)-2\varphi(a)\varphi(b)+\varphi(a)\varphi(b)\\ &= \varphi(ab)-\varphi(a)\varphi(b),\end{aligned}$$

and so $\varphi(ab)=\varphi(a)\varphi(b)$.

Freeness - examples

It turns out that $\varphi(aba) = \varphi(a^2)\varphi(b)$ yet

$$\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2,$$

whereas if a, b were to commute,

$$\varphi(abab) = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2).$$

Freeness in random matrices and the Haar measure

As mentioned, it does not suffice that the **A** matrices and the **B** matrices are independent in order to guarantee freeness between the corresponding variables a, b.

To guarantee freeness, one can "rotate" one of the matrices. It is a deep result in probability theory that for every "nice" group there is a unique measure that is invariant under translations. This is called the Haar measure.

In our case, let $\mathbf{U}$ be a unitary $n \times n$ matrix that is sampled according to the Haar measure on the group of unitary $n \times n$ matrices. Although the variables corresponding to $\mathbf{A}, \mathbf{B}$ are not free, the variables corresponding to $\mathbf{A}, \mathbf{U}\mathbf{B}\mathbf{U}^{\mathsf{T}}$ are!

Free convolution

Free probability theory provides strong tools for studying the sum of two free variables. Thus, studying random matrices of the form $\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{U}^T$ and more generally,

$$\sum_{i=1}^{d} \mathbf{U}_i \mathbf{A} \mathbf{U}_i^T.$$

Compare this with MSSb's distribution of d random matchings which can be written as

$$\sum_{i=1}^{d} \mathbf{\Pi}_i \mathbf{A} \mathbf{\Pi}_i^T,$$

where the $\mathbf{\Pi}_i$ are random permutation matrices (rather than uniformly distributed according to the Haar measure).

Free convolution and the R transform

Recall that a matching has half its eigenvalues -1 and half $+1$. Thus, the corresponding Cauchy transform is

$$\begin{aligned} G(z) &= \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) \\ &= \frac{1}{2} \cdot \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{z+1} \\ &= \frac{z}{z^2-1}. \end{aligned}$$

As it turns out, while G does not behave nicely under addition (that is, G_{a+b} has no apparent clean expression in terms of G_a, G_b), there is a related function, called the R-transform that does behave well.

Free convolution and the R transform

To compute the R transform corresponding to G, one first invert G with respect to composition, namely, find K such that

$$z = K(G(z)).$$

The R transform is then given by

$$R(z) = K(z) - \frac{1}{z}.$$

What is great about the R transform is that

$$R_{a+b}(z) = R_a(z) + R_b(z).$$

In our case, $G(z) = \frac{z}{z^2-1}$ and so to find $K(z)$ we need to solve

$$z = \frac{K(z)}{K(z)^2 - 1}$$

Free convolution and the R transform

Which gives

$$K(z) = \frac{1 + \sqrt{1 + 4z^2}}{2z}.$$

Thus,

$$R(z) = K(z) - \frac{1}{z} = \frac{\sqrt{1 + 4z^2} - 1}{2z}.$$

Therefore, the R-transform of the sum of d free copies is

$$R_d(z) = d \cdot R(z) = \frac{d}{2} \cdot \frac{\sqrt{1 + 4z^2} - 1}{z}.$$

Rolling back to K_d, we get

$$K_d(z) = R_d(z) + \frac{1}{z} = \frac{d\sqrt{1 + 4z^2} - (d - 2)}{2z}.$$

Free convolution and the R transform

Having computed

$$K_d(z) = R_d(z) + \frac{1}{z} = \frac{d\sqrt{1+4z^2} - (d-2)}{2z}$$

we again use the relation $z = K_d(G_d(z))$ to get

$$G_d(z) = \frac{d\sqrt{z^2 - 4(d-1)} - z(d-2)}{2(z^2 - d^2)}.$$

In order to get the density function of the sum of d copies at t, we need to take the imaginary part of $G(t + i\varepsilon)$ as $\varepsilon \to 0$.

Free convolution and the R transform

$$G_d(z) = \frac{d\sqrt{z^2 - 4(d-1)} - z(d-2)}{2(z^2 - d^2)}.$$

In order to get the density function of the sum of d copies at t, we need to take the imaginary part of $G(t + i\varepsilon)$ as $\varepsilon \to 0$.

If we only wish to know the support, we see that it must be the case that the $\sqrt{\cdot}$ is given a negative argument. That is, every t in the support satifies

$$t^2 < 4(d-1)$$

or, equivalently,

$$|t| < 2\sqrt{d-1}.$$

MSSb transform this "prediction" of free probability to an actual statement using their homemade finite free probability and properties of interlacing polynomials.