Weil Differentials

Unit 14

Gil Cohen

December 17, 2024

Overview

Weil Differentials

When discussing adeles, we proved that for every $\mathfrak{a} \in \mathcal{D}$,

$$\dim_{\mathsf{K}} \mathbb{A} \Big/ (\Lambda(\mathfrak{a}) + \mathsf{F}) = g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}).$$

To better understand the K-vector space

$$V = \mathbb{A} \Big/ (\Lambda(\mathfrak{a}) + \mathsf{F})$$

we will consider its functionals

$$\mathrm{Hom}_{\mathsf{K}}(V, \mathsf{K}) = \{\alpha : V \to \mathsf{K} \mid \alpha \text{ is K-linear}\}.$$

Equivalently, we will study K-linear maps from $\mathbb{A}$ to K that vanish on $\Lambda(\mathfrak{a}) + \mathsf{F}$.

Weil Differentials

Definition 1 (Weil differential)

Let F/K be a function field. A Weil differential is an element

$$\omega \in \mathrm{Hom}_K(\mathbb{A}, K)$$

that vanishes on $\Lambda(\mathfrak{a}) + F$ for some $\mathfrak{a} \in \mathcal{D}_{F/K}$.

The set of all Weil differentials of F/K is denoted by $\Omega = \Omega_{F/K}$.

The definition seems to have little to do with the more familiar notion of a differential. Namely, an operator d that "differentiate" functions having properties such as

$$d(f+g) = df + dg$$
$$d(fg) = f(dg) + g(df).$$

In the seminar part of the course you will get the chance to learn more about this connection. Still, we will explore this relation a bit now.

Overview

Weil Differentials and “ordinary” differentials

Definition 2

Let F/K be a function field. A map

$$\delta : F \to F$$

is a derivation of F/K if it is K-linear and it satisfies the product rule

$$\delta(uv) = u \cdot \delta(v) + v \cdot \delta(u)$$

for all $u, v \in F$.

Weil Differentials and "ordinary" differentials

Definition 3

An element $x \in \mathsf{F}$ is called a separating element of F/K provided that $\mathsf{F}/\mathsf{K}(x)$ is algebraic and separable.

Lemma 4

Let x be a separating element of F/K. Then, there exists a unique derivation

$$\delta_x : \mathsf{F} \to \mathsf{F}$$

of F/K s.t.

$$\delta_x(x) = 1.$$

δ_x is called the derivation with respect to x.

Weil Differentials and "ordinary" differentials

Definition 5

Let
$$\mathrm{Der}_{\mathsf{F}} = \{\eta : \mathsf{F} \to \mathsf{F} \mid \eta \text{ is a derivation of } \mathsf{F}/\mathsf{K}\}.$$

Note that $\mathrm{Der}_{\mathsf{F}}$ is an F-vector space:

$$(\eta_1 + \eta_2)(z) = \eta_1(z) + \eta_2(z)$$
$$(u\eta)(z) = u \cdot \eta(z).$$

$\mathrm{Der}_{\mathsf{F}}$ is called the the vector space of derivations of F/K.

Lemma 6

Let x be a separating element of F/K. Then, for each $\eta \in \mathrm{Der}_{\mathsf{F}}$ we have that

$$\eta = \eta(x) \cdot \delta_x.$$

In particular,

$$\dim_{\mathsf{F}} \mathrm{Der}_{\mathsf{F}} = 1.$$

Weil Differentials and "ordinary" differentials

Definition 7

On the set

$$Z = \{(u, x) \in \mathsf{F} \times \mathsf{F} \mid x \text{ is a separating element}\}$$

define the relation

$$(u, x) \sim (v, y) \iff v = u \cdot \delta_y(x).$$

$\sim$ is an equivalence relation. We write

$$u\,dx$$

for the class containing (u, x) and call it a differential.

Weil Differentials and "ordinary" differentials

Definition 8

Let

$$\Delta_{\mathsf{F}} = \{u\,dx \mid x \text{ is a separating element}\}$$

be the set of all differentials of F/K.

It turns out we can add up differentials $u\,dx$, $v\,dy$ as follows: choose a separating element z, and use the chain rule to write

$$u\,dx = (u \cdot \delta_z(x))\,dz,$$
$$v\,dy = (v \cdot \delta_z(y))\,dz,$$

and define

$$u\,dx + v\,dy = (u \cdot \delta_z(x) + v \cdot \delta_z(y))\,dz.$$

Likewise,

$$w \cdot (u\,dx) = (wu)\,dx \in \Delta_{\mathsf{F}},$$

and so Δ_{F} is an F-vector space.

Weil Differentials and "ordinary" differentials

Definition 9

Define the map

$$d : \mathsf{F} \to \Delta_{\mathsf{F}}$$
$$t \mapsto dt$$

with the understanding that $dt = 0$ for t non-separating.

Lemma 10

Let $z \in \mathsf{F}$ be a separating element. Then, $dz \neq 0$, and every differential $\omega \in \Delta_{\mathsf{F}}$ can be written in the form

$$\omega = u\, dz$$

for some $u \in \mathsf{F}$. In particular,

$$\dim_{\mathsf{F}} \Delta_F = 1.$$

Moreover, d is a derivation (though to Δ_{F} rather than to F).

Weil Differentials and “ordinary” differentials

Since

$$\dim_{\mathsf{F}} \Delta_F = 1$$

we can define the quotient of differentials ω_1 and $\omega_2 \neq 0$ by

$$\frac{\omega_1}{\omega_2} = u \in \mathsf{F},$$

where u is the unique element in F s.t. $\omega_1 = u\omega_2$. In particular,

$$\delta_z(y) = \frac{dy}{dz}.$$

The chain rule, for example, takes the form

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz}.$$

Overview

Technicality

Let F/K be a function field. Let V be an F-vector space and W a K-vector space.

We know that $\mathrm{Hom}_{\mathrm{K}}(V, W)$ is a K-vector space. Indeed, if

$$\varphi_1, \varphi_2 : V \to W$$

are K-linear then so is their sum $\varphi_1 + \varphi_2$ and $a\varphi_1$ for every $a \in \mathrm{K}$.

That holds true even if V is a K-vector space.

As V is an F-vector space, $\mathrm{Hom}_{\mathrm{K}}(V, W)$ is also an F-vector space. Indeed, for $a \in \mathrm{F}$ and $\varphi \in \mathrm{Hom}_{\mathrm{K}}(V, W)$, we define

$$(a\varphi)(v) = \varphi(av).$$

One can show $a\varphi \in \mathrm{Hom}_{\mathrm{K}}(V, W)$. E.g., for $b \in \mathrm{K}$ and $v \in V$,

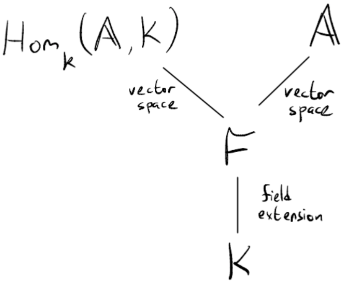
$$(a\varphi)(bv) = \varphi(abv) = b \cdot \varphi(av) = b \cdot (a\varphi)(v).$$

Technicality

Moreover, if $a \in \mathsf{K}$ then

$$(a\varphi)(v) = \varphi(av) = a \cdot \varphi(v)$$

and so the multiplication by an element of F as we have just defined it, extends the good old multiplication of an element by K. In particular,



Weil Differentials

Definition 11

Let F/K be a function field and $\mathfrak{a} \in \mathcal{D}_{\mathsf{F}/\mathsf{K}}$. We define

$$\Omega(\mathfrak{a}) = \left\{\omega \in \Omega_{\mathsf{F}/\mathsf{K}} \mid \omega(\Lambda(\mathfrak{a}) + \mathsf{F}) = 0\right\}.$$

Claim 12

$\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{D}$ and $x \in \mathsf{F}^{\times}$,

1. $\mathfrak{a} \leq \mathfrak{b} \implies \Omega(\mathfrak{b}) \subseteq \Omega(\mathfrak{a})$.
2. $\Omega(\mathfrak{a}) + \Omega(\mathfrak{b}) \subseteq \Omega(\min(\mathfrak{a}, \mathfrak{b}))$.
3. $\Omega(\mathfrak{a}) \cap \Omega(\mathfrak{b}) = \Omega(\max(\mathfrak{a}, \mathfrak{b}))$.
4. $x\Omega(\mathfrak{a}) = \Omega(\mathfrak{a} + (x))$.
5. $\Omega = \cup_{\mathfrak{a} \in \mathcal{D}} \Omega(\mathfrak{a})$.

Left as an exercise.

Claim 13

$\forall \mathfrak{a} \in \mathcal{D}$, $\Omega(\mathfrak{a})$ is a subspace of $\mathrm{Hom}_{\mathsf{K}}(\mathbb{A}, \mathsf{K})$ as a K-vector space.

Proof.

$\Omega(\mathfrak{a})$ is clearly closed under addition. Moreover, for $x \in \mathsf{K}^{\times}$,

$$x\Lambda(\mathfrak{a}) = \Lambda(\mathfrak{a} - (x)) = \Lambda(\mathfrak{a}),$$

and so

$$\omega \in \Omega(\mathfrak{a}) \quad \Longrightarrow \quad \begin{aligned} (x\omega)(\Lambda(\mathfrak{a}) + \mathsf{F}) &= \omega(x(\Lambda(\mathfrak{a}) + \mathsf{F})) \\ &= \omega(\Lambda(\mathfrak{a}) + \mathsf{F}) \\ &= 0. \end{aligned}$$

□

Definition 14

For a divisor $\mathfrak{a}$, we define the index of specialty of $\mathfrak{a}$ by

$$\delta(\mathfrak{a}) = \dim_K \Omega(\mathfrak{a}),$$

also noting that

$$\begin{aligned}\delta(\mathfrak{a}) &= \dim_K \mathbb{A} \big/ (\Lambda(\mathfrak{a}) + F) \\ &= g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}).\end{aligned}$$

Weil Differentials

Claim 15

$$\Omega = \bigcup_{\mathfrak{a} \in \mathcal{D}} \Omega(\mathfrak{a})$$

is an F-vector space.

Proof.

Take $\omega \in \Omega$ and $x \in \mathsf{F}^{\times}$. Let $\mathfrak{a} \in \mathcal{D}$ s.t. $\omega \in \Omega(\mathfrak{a})$. Then,

$$\begin{aligned}
(x\omega)(\Lambda(\mathfrak{a} + (x)) + \mathsf{F}) &= \omega(x(\Lambda(\mathfrak{a} + (x)) + \mathsf{F})) \\
&= \omega(\Lambda(\mathfrak{a}) + \mathsf{F}) \\
&= 0.
\end{aligned}$$

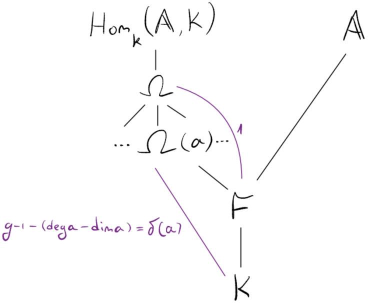
Take $\omega_1, \omega_2 \in \Omega$. Then, $\omega_1 \in \Omega(\mathfrak{a}_1)$, $\omega_2 \in \Omega(\mathfrak{a}_2)$, and so by Claim 12,

$$\omega_1 + \omega_2 \in \Omega(\min(\mathfrak{a}_1, \mathfrak{a}_2)) \subseteq \Omega.$$

Weil Differentials

Theorem 16

$\dim_F \Omega = 1.$



Informally, and inaccurately, if we think of Ω as differentials $\Omega = \{dx \mid x \in F\}$ then Theorem 16 is to be expected as

$$dy = \frac{dy}{dx} \cdot dx.$$

Weil Differentials

Proof.

Let $\omega_1, \omega_2 \in \Omega \setminus \{0\}$. We want to find $x \in \mathsf{F}^\times$ s.t. $\omega_2 = x\omega_1$. As

$$\Omega(\mathfrak{a}) + \Omega(\mathfrak{b}) \subseteq \Omega(\min(\mathfrak{a}, \mathfrak{b})),$$

we my assume that $\omega_1, \omega_2 \in \Omega(\mathfrak{b})$ for some $\mathfrak{b} \in \mathcal{D}$.

Take $\mathfrak{a} \in \mathcal{D}$, $\mathfrak{a} < 0$, with a "sufficiently low" degree $d = \deg \mathfrak{a}$.

As $\mathfrak{a} < 0$ we have that

$$\dim \mathfrak{a} = \dim_{\mathsf{K}} \mathcal{L}(\mathfrak{a}) = 0,$$

and so for a sufficiently large $|d|$,

$$\begin{aligned} \delta(\mathfrak{a}) &= \dim_{\mathsf{K}} \Omega(\mathfrak{a}) \\ &= g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}) \\ &= g - 1 - d > 0. \end{aligned}$$

Weil Differentials

Proof.

For $i = 1, 2$ define the map

$$\begin{aligned} \mathsf{F} &\to \Omega \\ x &\mapsto x\omega_i. \end{aligned}$$

These are injective K-linear maps. Further, each induces a map

$$T_i : \mathcal{L}(\mathfrak{b} - \mathfrak{a}) \to \Omega(\mathfrak{a}).$$

Indeed, if $x \in \mathcal{L}(\mathfrak{b} - \mathfrak{a})$ then $(x) + \mathfrak{b} \geq \mathfrak{a}$. Thus,

$$x\omega_i \in x\Omega(\mathfrak{b}) = \Omega((x) + \mathfrak{b}) \subseteq \Omega(\mathfrak{a}).$$

Proof.

By Riemann's Theorem,

$$\begin{aligned} g-1 &\geq \deg(\mathfrak{b}-\mathfrak{a}) - \dim(\mathfrak{b}-\mathfrak{a}) \\ &= -d + \deg\mathfrak{b} - \dim \operatorname{Im} T_i. \end{aligned}$$

Thus, by taking $|d|$ large enough,

$$\begin{aligned} \dim \operatorname{Im} T_i &\geq -d + \deg\mathfrak{b} - g + 1 \\ &> \frac{1}{2}(g-1-d) \\ &= \frac{\delta(\mathfrak{a})}{2}, \end{aligned}$$

as indeed

$$\delta(\mathfrak{a}) = g - 1 - (\deg\mathfrak{a} - \dim\mathfrak{a}) = g - 1 - d.$$

Weil Differentials

Proof.

As $\delta(\mathfrak{a}) = \dim_K \Omega(\mathfrak{a})$ and since

$$\dim_K \operatorname{Im} T_1,\ \dim_K \operatorname{Im} T_2 > \frac{\delta(\mathfrak{a})}{2},$$

the two subspaces intersect non trivially.

Therefore, $\exists x_1, x_2 \in F^\times$ s.t.

$$x_1\omega_1 = x_2\omega_2$$

which concludes the proof. □

Weil Differentials

Consider the space of adeles which are everywhere defined ("holomorphic adeles" if you will)

$$\Lambda(0) = \{\alpha \in \mathbb{A} \mid v_{\mathfrak{p}}(\alpha) \geq 0\},$$

and that

$$\Omega(0) = \{\omega \in \Omega \mid \omega(\Lambda(0) + \mathsf{F}) = 0\}.$$

We have the following characterization of the genus as the index of specialty of the zero divisor.

Claim 17

$\delta(0) = \dim_{\mathsf{K}} \Omega(0) = g.$

Proof.

As $\mathcal{L}(0) = \mathsf{K}$,

$$\delta(0) = g - 1 - (\deg 0 - \dim 0) = g.$$

Weil Differentials

Recall that

$$\mathfrak{a} \text{ large} \implies \Lambda(\mathfrak{a}) \text{ large} \implies \Omega(\mathfrak{a}) \text{ small.}$$

Claim 18

$$\Omega(\mathfrak{a}) \neq \{0\} \implies \dim \mathfrak{a} \leq g.$$

Proof.

Take $0 \neq \omega \in \Omega(\mathfrak{a})$. Consider the K-monomorphism

$$\mathcal{L}(\mathfrak{a}) \to \Omega$$
$$x \mapsto x\omega$$

Now,

$$x\omega \in x\Omega(\mathfrak{a}) = \Omega(\mathfrak{a} + (x)) \subseteq \Omega(0).$$

Thus, by Claim 17,

$$\dim \mathfrak{a} = \dim_K \mathcal{L}(\mathfrak{a}) \leq \dim_K \Omega(0) = g.$$

Overview

Canonical divisors

Recall that $\omega \in \Omega(\mathfrak{b}) \iff \omega(\Lambda(\mathfrak{b}) + \mathsf{F}) = 0$, and so if $\omega \in \Omega(\mathfrak{b})$ then

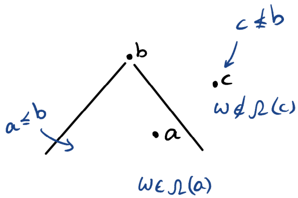
$$\mathfrak{a} \leq \mathfrak{b} \implies \omega \in \Omega(\mathfrak{a}).$$

Theorem 19

For every $0 \neq \omega \in \Omega$ there exists a unique $\mathfrak{b} \in \mathcal{D}$ satisfying

$$\omega \in \Omega(\mathfrak{a}) \iff \mathfrak{a} \leq \mathfrak{b}.$$

This unique divisor $\mathfrak{b}$ is denoted by (ω).



Canonical divisors

Proof.

Consider $\mathfrak{a} \in \mathcal{D}$ s.t. $\omega \in \Omega(\mathfrak{a})$. By Claim 18, $\dim \mathfrak{a} \leq g$.

By Riemann's Theorem,

$$\deg \mathfrak{a} \leq 2g - 1.$$

Thus, we can take a divisor of maximal degree $\mathfrak{b}$ s.t. $\omega \in \Omega(\mathfrak{b})$. Take any $\mathfrak{a} \in \mathcal{D}$ s.t. $\omega \in \Omega(\mathfrak{a})$. Then,

$$\omega \in \Omega(\mathfrak{a}) \cap \Omega(\mathfrak{b}) = \Omega(\max(\mathfrak{a}, \mathfrak{b})).$$

But by the maximality of the degree of $\mathfrak{b}$,

$$\deg \mathfrak{b} \geq \deg \max(\mathfrak{a}, \mathfrak{b}),$$

and so

$$\mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b}) \geq \mathfrak{a}.$$

Uniqueness is obvious. □

Canonical divisors

Recall that Ω is an F-vector space via $(x\omega)(\alpha) = \omega(x\alpha)$.

Claim 20

For $0 \neq \omega \in \Omega$ and $x \in F^{\times}$,

$$(x\omega) = (x) + (\omega).$$

Proof.

By Theorem 19,

$$\begin{aligned} x\omega \in \Omega(\mathfrak{a}) \quad &\iff \quad \omega \in x^{-1}\Omega(\mathfrak{a}) = \Omega(\mathfrak{a} - (x)) \\ &\iff \quad (\omega) \geq \mathfrak{a} - (x) \\ &\iff \quad \mathfrak{a} \leq (x) + (\omega). \end{aligned}$$

But we also have, by Theorem 19, that

$$x\omega \in \Omega(\mathfrak{a}) \quad \iff \quad \mathfrak{a} \leq (x\omega),$$

and $(x\omega)$ is the unique such divisor. Thus, $(x\omega) = (x) + (\omega)$. □

Canonical divisors

Definition 21

A divisor of the form (ω) for $\omega \in \Omega$ is called canonical. The set of all canonical divisors is denoted by $\mathcal{W}$.

Claim 22

$\mathcal{W}$ is an element of $\mathsf{C} = \mathcal{D}/\mathcal{P}$.

This explains why we call a canonical divisor "canonical". Perhaps a better name would have been a canonical divisor class.

Canonical divisors

Proof.

Take $0 \neq \omega \in \Omega$. By Theorem 16,

$$\Omega = \{x\omega \mid x \in \mathsf{F}\},$$

and so, by Claim 20,

$$\begin{aligned}
\mathcal{W} &= \{(\omega') \mid \omega' \in \Omega\} \\
&= \{(x\omega) \mid x \in \mathsf{F}\} \\
&= \{(x) + (\omega) \mid x \in \mathsf{F}\} \\
&= (\omega) + \{(x) \mid x \in \mathsf{F}\} \\
&= (\omega) + \mathcal{P}.
\end{aligned}$$

□