AG codes - Spring 2022
Problem Set 04

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Problem 1. *Let F/K be a function field with $g = 0$. Prove that every divisor a of F/K, with* $\deg(a) = 0$ *is principle.*

Problem 2. *Let F/K be a function field. Let $x \in F \setminus K$. Show that $\mathcal{L}(n(x)_\infty) \cap K(x) = \{f(x) \mid f \in K[X]$ s.t. $\deg(f) \leq n\}$.*

Problem 3. *Let F_1/K_1, F_2/K_2 be two function fields. Denote by $\mathbb{P}_1, \mathbb{P}_2$ there prime divisors set, and by $\mathcal{D}_1, \mathcal{D}_2$ their divisor groups, respectively. Let $\sigma : F_1 \to F_2$ satisfying $\sigma(K_1) = K_2$.*

- *(a) Show that there is a on to one correspondence $\tilde{\sigma} : \mathbb{P}_1 \to \mathbb{P}_2$ given by $v_{\tilde{\sigma}(p)}(x) = v_p(\sigma^{-1}(x))$.*
- *(b) Show that we can extend $\tilde{\sigma} : \mathcal{D}_1 \to \mathcal{D}_2$ to be a group isomorphism.*
- *(c) Prove that for every $x \in F_1$, $\tilde{\sigma}((x)) = (\sigma(x))$, and $\tilde{\sigma}((x)_\infty) = (\sigma(x))_\infty$.*
- *(d) Prove that for every divisor $\mathfrak{a} \in \mathcal{D}_1$, $\mathcal{L}(\tilde{\sigma}(\mathfrak{a})) = \sigma(\mathcal{L}(\mathfrak{a}))$.*

Problem 4. *Recall the function field generated by the hermitian curve we saw in class:*

$$F/\mathbb{F}_{p^2} = Frac(\mathbb{F}_{p^2}[x, y]/(Tr(y) = Norm(x))$$

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- *(a) Explicitly define the Goppa code this function field (choose your places and describe the encoding).*
- *(b) Calculate the rate, distance and alphabet size of the code.*
- *(c) Compare it to Reed Solomon and the two dimensional reed-Muller code.*

Use the fact that the genus of F is $\frac{p(p-1)}{2}$.

Problem 5. *Let F/K be a finite extension of E/K, with K perfect. Show that, for every algebraic separable extension L/K,*

$$[F : E] = [FL : EL].$$

Problem 6.

Definition. *Let $P \in \mathbb{P}_F$.*

- *(a) For $x \in F$ let $\iota_P(x) \in \mathcal{A}_F$ be the adele whose P-component is x, and all other components are 0.*
- *(b) For a Weil differential $\omega \in \Omega_F$ we define its local component $\omega_P : F \to K$ by $\omega_P(x) := \omega(\iota_P(x))$. Clearly ω_P is a K-linear mapping.*

Let $\omega \in \Omega_F$ and $\alpha = (\alpha_P) \in \mathcal{A}_F$. Prove that $\omega_P(\alpha_P) \neq 0$ for at most finitely many places P, and

$$\omega(\alpha) = \sum_{P \in \mathbb{P}_F} \omega_P(\alpha_P).$$

In particular

$$\sum_{P \in \mathbb{P}_F} \omega_P(1) = 0.$$

Guidance: denote $W = (w) \in \mathcal{D}_F$., and let $S \subseteq \mathbb{P}_F$ be finite s.t.

$$v_P(W) = 0 \text{ and } v_P(\alpha_P) \geq 0 \text{ for all } P \notin S$$

And define $\beta = (\beta_P) \in \mathcal{A}_F$:

$$\beta_P = \begin{cases} \alpha_P & P \notin S \\ 0 & \text{otherwise.} \end{cases}$$