Expander Graphs

Following Vadhan, Chapter 4

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Outline

Vertex expansion

Definition

Let $G = (V, E)$ be an undirected graph. For $S \subseteq V$ we define the neighborhood of S by

$$\Gamma(S) = \{u \in V \mid \exists v \in S \; uv \in E\}.$$

Definition

An undirected graph $G = (V, E)$ is a (k, a)-vertex expander if for every $S \subset V$ of size at most k it holds that $|\Gamma(S)| \geq a|S|$.

Vertex expansion

Theorem

For every $d \geq 3$ there is $\alpha > 0$ such that the following holds. For every integer $n \geq 1$ there exists a d-regular undirected graph on n vertices that is an $(\alpha n, d - 1.01)$-vertex expander.

Spectral expanders

Definition

Let $G = (V, E)$ be an undirected graph. The spectral gap of G is defined by

$$\gamma(G) = \omega_1(G) - \omega(G) = 1 - \omega(G),$$

where, recall, $\omega(G) = \max(\omega_2(G), -\omega_n(G))$.

We say that G is a γ-spectral expander if $\gamma(G) \geq \gamma$.

Recall that the spectral gap is related to the rate of convergence of a random walk as

$$\omega(G) = \max_{\mathbf{p}} \frac{\|\mathbf{Wp} - \mathbf{u}\|}{\|\mathbf{p} - \mathbf{u}\|}$$

Spectral expanders

Theorem

If G is a γ-spectral expander then it is an $(\frac{n}{2}, 1+\gamma)$-vertex expander.

To prove the theorem, we define

Definition

Let $\mathbf{p}$ be a distribution. The collision probability of $\mathbf{p}$ is the probability two independent samples from $\mathbf{p}$ are equal. Namely,

$$\mathrm{CP}(\mathbf{p}) = \sum_x \mathbf{p}(x)^2.$$

Spectral expanders

Lemma

For every probability distribution $\mathbf{p} \in [0,1]^n$,

1. $\mathrm{CP}(\mathbf{p}) = \|\mathbf{p}\|^2 = \|\mathbf{p} - \mathbf{u}\|^2 + \frac{1}{n}$.
2. $\mathrm{CP}(\mathbf{p}) \geq \frac{1}{|\sup(\mathbf{p})|}$.

Spectral expanders

We now prove the theorem, restated below.

Theorem

If G is a γ-spectral expander then it is an $(\frac{n}{2}, 1+\gamma)$-vertex expander.

Extra space for the proof

Edge expansion

Definition

An undirected graph $G = (V, E)$ is a (k, ε)-edge expander if for every $S \subseteq V$ of size $|S| \leq k$,

$$|\partial(S)| \geq \varepsilon d|S|.$$

Recall that the conductance of S for a d-regular graphs is

$$\phi(S) = \frac{|\partial(S)|}{\min(d(S), d(V \setminus S))} = \frac{|\partial(S)|}{d \min(|S|, |V \setminus S|).}$$

Hence, for simplicity, focusing on $k = \frac{n}{2}$, in an $(\frac{n}{2}, \varepsilon)$-edge expander every set of size at most $\frac{n}{2}$ has conductance at least ε.

Edge expansion vs spectral expansion

By Cheeger's inequality

$$\frac{\nu_2}{2} \leq \phi(G) \leq \sqrt{2\nu_2}.$$

Now,

$$1 - \gamma = \omega = \max(\omega_2, -\omega_n) \geq \omega_2 = 1 - \nu_2,$$

and so $\phi(G) \geq \frac{\nu_2}{2} \geq \frac{\gamma}{2}$.

For the other direction, we need to make sure $-\omega_n \leq \omega_2$. One way is to add sufficiently many self loops so to guarantee $\omega_n \geq 0$.

Spectral norm

Definition

Let $\mathbf{A}$ be a real matrix. The spectral norm of $\mathbf{A}$, denoted by $\|\mathbf{A}\|$, is given by

$$\|\mathbf{A}\| = \max_{0 \neq \mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

Geometrically, $\|\mathbf{A}\|$ measures the largest "stretch" $\mathbf{A}$ can have.

Spectral norm

Lemma

The spectral norm of $\mathbf{A}$*, equals to the square root of the largest eigenvalue of* $\mathbf{A}^T\mathbf{A}$*. In particular, when* $\mathbf{A}$ *is symmetric,*

$$\|\mathbf{A}\| = \max\{|\alpha| : \alpha \in \mathrm{Spec}(\mathbf{A})\}.$$

Spectral norm

Lemma

We have the following properties of the spectral norm.

- *Subadditivity:* $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.
- *Submultiplactivity:* $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.
- $\|a\mathbf{A}\| \leq |a| \; \|\mathbf{A}\|$.

Another view on spectral expanders

Lemma

Let $G = (V, E)$ be an undirected regular graph. Then, G is a γ-spectral expander if and only if

$$\mathbf{W}_G = \gamma \mathbf{J} + (1 - \gamma)\mathbf{E},$$

where $\mathbf{J}$ stands for the $n \times n$ all $\frac{1}{n}$ matrix, and $\|\mathbf{E}\| \leq 1$.

Extra space for the proof

Another view on spectral expanders

It is sometimes more convenient to decompose $\mathbf{W}_G$ as $\mathbf{W}_G = \mathbf{J} + \mathbf{E}$, where $\|\mathbf{E}\| \leq \omega$.

Outline

The expander mixing lemma

Given two sets S, T of vertices, we denote

$$e(S, T) = \{uv \in E \mid u \in S, v \in T\}.$$

Lemma (The expander mixing lemma)

Let G be a d-regular $\gamma = 1 - \omega$ spectral expander on n vertices. Let $S, T \subseteq V$ be sets of density α, β respectively. Then,

$$\left| \frac{|e(S, T)|}{nd} - \alpha\beta \right| \leq \omega\sqrt{\alpha(1-\alpha)\beta(1-\beta)}.$$

Extra space for the proof

Extra space for the proof

Hitting property of expander walks

Theorem

Let $G = (V, E)$ be a d-regular $\gamma = 1 - \omega$ spectral expander. Let $v_1, \ldots, v_t$ be a random walk in which v_1 is sampled uniformly at random from V. Then, for every $B \subseteq V$ having density μ,

$$\Pr\left[\{v_1, \ldots, v_t\} \subseteq B\right] \leq (\mu + \omega)^t.$$

Hitting property of expander walks

Claim

Let **P** *be the diagonal matrix indicating* B*. Then,*

$$\begin{aligned}\Pr\left[\{v_1, \ldots, v_t\} \subseteq B\right] &= \|(\mathbf{PW})^{t-1}\mathbf{Pu}\|_1 \\ &= \|(\mathbf{PWP})^{t-1}\mathbf{Pu}\|_1.\end{aligned}$$

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Extra space for the proof

Hitting property of expander walks

Claim

$$\|\mathbf{PJP}\| \leq \mu + \omega.$$

Extra space for the proof

Outline

Error reduction via expander random walks

Suppose we have a one-sided error randomized algorithm that uses r random bits and has constant error probability, say, $\frac{1}{2}$. Our goal is to reduce the error to ε with low cost in randomness.

Naively, we can run the algorithm $\log(1/\varepsilon)$ times, using fresh randomness each time, and return the AND (or OR) of the results. The randomness complexity is $r \cdot \log(1/\varepsilon)$.

Using expanders, we only need $r + O(\log(1/\varepsilon))$ random bits!

We remark that this savings can be obtained using pairwise independent distributions as well. Then, however, there is a $(1/\varepsilon)^{O(1)}$ blowup in time complexity.

A similar method works also for two-sided error, where the analysis is based on the expander Chernoff bound.

Extra space for the proof

Ramanujan graphs

A natural question is how large can we make γ (equivalently, small ω) as a function of d? In the problem set, you will prove the Alon-Boppana bound

$$\omega \geq \frac{2\sqrt{d-1}}{d} - \varepsilon(n),$$

where $\varepsilon(n) \to 0$. Remarkably, this is tight: there are graphs with

$$\omega \leq \frac{2\sqrt{d-1}}{d}.$$

Graphs meeting this bound are called Ramanujan graphs.

Interestingly, random d-regular graphs achieve, w.h.p, "only" $\omega \leq \frac{2\sqrt{d-1}}{d} + \varepsilon(n)$, with $\varepsilon(n) \to 0$.