# Abstracting the Notion of a Point

Unit 2

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# Overview

In this unit we will informally discuss how to abstract the notion of a point. The formal treatment is given in the next several units.

1. Points and maximal ideals
2. Valuation rings
3. Valuations
4. Places
5. Further remarks

# The line

Let F be an algebraically closed field. Every maximal ideal of $F[x]$ is of the form $\langle x - \mathfrak{p} \rangle$, and vice versa. Indeed,

On the one hand, $F[x]/\langle x - \mathfrak{p} \rangle \cong F$ is a field. Indeed, consider the ring homomorphism $\psi : F[x] \to F$ that maps $f(x) \mapsto f(\mathfrak{p})$. We have that $\ker \psi = \langle x - \mathfrak{p} \rangle$, and so by the first homomorphism theorem $F[x]/\langle x - \mathfrak{p} \rangle \cong F$. Thus, $\langle x - \mathfrak{p} \rangle$ is a maximal ideal.

On the other hand, F is a field $\implies$ $F[x]$ is a PID. As F is algebraically closed, $\langle f(x) \rangle$ is maximal $\iff$ $\deg f(x) = 1$.

| Geometry | Algebra |
|---|---|
| $F$ $(\mathbb{A}^1(F))$ | $F[x]$ |
| point $p \in F$ | maximal ideal $\langle x - p \rangle$ |

# The plane

Again assume that F is algebraically closed.

Every maximal ideal of F$[x, y]$ corresponds to a point $\mathfrak{p} \in$ F $\times$ F–it is of the form $\langle x - \mathfrak{p}_1, y - \mathfrak{p}_2 \rangle$–and vice versa. This is a special case of Hilbert's Nullstellensatz (in its weak form). We plan on proving this in the recitation.

| Geometry | Algebra |
| --- | --- |
| $F \times F$ $(\mathbb{A}^2(F))$ | $F[x, y]$ |
| point $p = (p_1, p_2) \in F \times F$ | maximal ideal $\langle x - p_1, y - p_2 \rangle$ |

# A plane curve

Again assume that $\mathbb{F}$ is algebraically closed and let $f(x, y) \in \mathbb{F}[x, y]$ be irreducible. Define

$$Z_f = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid f(x, y) = 0\}.$$

A corollary of Hilbert's Nullstellensatz implies that points on the curve $Z_f$ are in bijection with the maximal ideals of the ring

$$C_f = \mathbb{F}[x, y] \big/ \langle f(x, y) \rangle.$$

| Geometry | Algebra |
|---|---|
| $Z_f = \{ p \in F \times F \mid f(p) = 0 \}$ | $C_f = F[x,y] / \langle f(x,y) \rangle$ |
| point $p \in Z_f$ | maximal ideal $\langle x - p_1, y - p_2 \rangle$ |

# Overview

# Valuation rings

Another way of attaching an algebraic object to a point is to consider the set of rational functions that can be evaluated at the point.

Again, assume that F is algebraically closed. For $\mathfrak{p} \in \mathsf{F}$ let

$$\mathcal{O}_{\mathfrak{p}} = \left\{ \frac{g(x)}{h(x)} \;\middle|\; g(x), h(x) \in \mathsf{F}[x] \text{ coprime and } h(\mathfrak{p}) \neq 0 \right\} \subseteq \mathsf{F}(x).$$

$\mathcal{O}_{\mathfrak{p}}$ determines $\mathfrak{p}$. Furthermore, $\mathcal{O}_{\mathfrak{p}}$ is a ring with the following special property:

$$\forall f(x) \in \mathsf{F}(x) \quad f(x) \in \mathcal{O}_{\mathfrak{p}} \text{ or } \frac{1}{f(x)} \in \mathcal{O}_{\mathfrak{p}}.$$

For those with background in commutative algebra, note that

$$\mathcal{O}_{\mathfrak{p}} = \mathsf{F}[x]_{\langle x - \mathfrak{p} \rangle}.$$

# Valuation rings

Is there another ring, with fraction field $\mathsf{F}(x)$, satisfying the property above?

Consider

$$\mathcal{O}_\infty = \left\{ \frac{g(x)}{h(x)} \;\middle|\; g(x), h(x) \in \mathsf{F}[x],\ h(x) \neq 0 \text{ and } \deg g(x) \leq \deg h(x) \right\}.$$

Geometrically, this ring corresponds to the "point at infinity". Indeed, the algebraic perspective is compatible with the so-called projective geometry rather than the affine geometry.

# Valuation rings and their maximal ideals

Note that

$$\mathcal{O}_{\mathfrak{p}} = \left\{ \frac{g(x)}{h(x)} \;\middle|\; g(x), h(x) \in \mathsf{F}[x] \text{ coprime and } h(\mathfrak{p}) \neq 0 \right\} \subseteq \mathsf{F}(x).$$

has a maximal ideal

$$\mathfrak{m}_{\mathfrak{p}} = \left\{ \frac{g(x)}{h(x)} \in \mathcal{O}_{\mathfrak{p}} \;\middle|\; g(\mathfrak{p}) = 0 \right\}.$$

Moreover, a maximal ideal of $\mathcal{O}_\infty$ is given by

$$\mathfrak{m}_{\infty} = \left\{ \frac{g(x)}{h(x)} \;\middle|\; g(x), h(x) \in \mathsf{F}[x],\ h(x) \neq 0 \text{ and } \deg g(x) < \deg h(x) \right\}.$$

As we will later prove, $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal in $\mathcal{O}_{\mathfrak{p}}$, and $\mathfrak{m}_\infty$ is the unique maximal ideal in $\mathcal{O}_\infty$. Further, $\mathfrak{m}$ determines $\mathcal{O}$.

# Overview

# Valuations

For an algebraically closed field F and $\mathfrak{p} \in \mathsf{F}$ consider the function

$$\upsilon_{\mathfrak{p}} : \mathsf{F}(x)^{\times} \to \mathbb{Z}$$

that counts the multiplicity of $\mathfrak{p}$ as a root or a pole of a given $f(x) \in \mathsf{F}(x)^{\times}$. Namely, if we write

$$f(x) = (x - \mathfrak{p})^r \frac{g(x)}{h(x)}$$

for coprime $g(x), h(x) \in \mathsf{F}[x]$ with $g(\mathfrak{p}), h(\mathfrak{p}) \neq 0$ then $\upsilon_{\mathfrak{p}}(f) = r$.

**Exercise.** For every $\mathfrak{p} \in \mathsf{F}$ determine

$$\upsilon_{\mathfrak{p}} \left( \frac{x^2 - 1}{x^3} \right).$$

We extend $\upsilon_{\mathfrak{p}}$ to $\mathsf{F}(x)$ by setting $\upsilon_{\mathfrak{p}}(0) = \infty$ with the understanding that $\infty > \mathbb{Z}$.

# Valuations

Note that $v_{\mathfrak{p}}$ determines and is determined by $\mathcal{O}_{\mathfrak{p}}$. Indeed,

$$\mathcal{O}_{\mathfrak{p}} = \{f(x) \in \mathsf{F}(x) \mid v_{\mathfrak{p}}(f) \geq 0\}.$$

Further, $\mathcal{O}_{\mathfrak{p}}$ determines $\mathfrak{p}$ which, in turn, determines $v_{\mathfrak{p}}$.

We will later call $v_{\mathfrak{p}}$ a valuation. The valuation that corresponds to $\mathcal{O}_{\infty}$ is given by

$$v_{\infty}\left(\frac{g(x)}{h(x)}\right) = \deg h(x) - \deg g(x).$$

E.g., $v_{\infty}(\frac{1}{x}) = 1$.

**Exercise.** Determine

$$v_{\infty}\left(\frac{x^2-1}{x^3}\right).$$

# Valuations

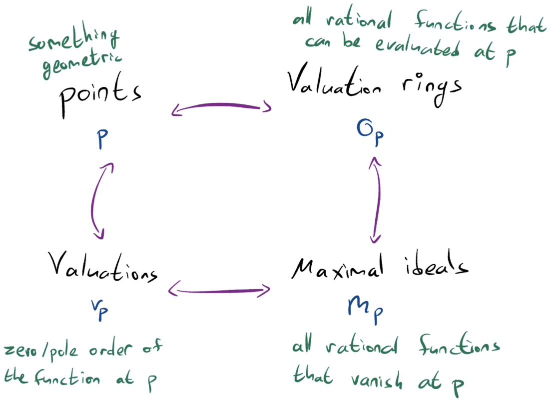
Note that a valuation $v$ (either $v_\infty$ or any of the $v_\mathfrak{p}$) satisfies the following properties. For every $f(x), g(x) \in \mathsf{F}(x)$

- $v(f) = \infty \iff f = 0$
- $v(fg) = v(f) + v(g)$
- $v(f + g) \geq \min(v(f), v(g))$

This will be the defining property of an abstract valuation in the sequel.

# So far

So far we informally suggested three ways of abstracting the notion of a point.




The fourth is given by the actual evaluation.

# Overview

# Places

Let F be an algebraically closed field. What does it mean to evaluate $f(x) = \frac{g(x)}{h(x)} \in \mathsf{F}(x)$ at $\mathfrak{p}$? (assuming $g(x), h(x)$ are coprime).

A computer science point of view is something like:

1. Substitute $\mathfrak{p}$ for $x$ in $h(x)$ to get $h(\mathfrak{p})$
2. If $h(\mathfrak{p}) = 0$ return $\infty$
3. Substitute $\mathfrak{p}$ for $x$ in $g(x)$ to get $g(\mathfrak{p})$
4. Return $\frac{g(\mathfrak{p})}{h(\mathfrak{p})} \in \mathsf{F}$.

An algebraic point of view would be

1. If $f(x) \notin \mathcal{O}_{\mathfrak{p}}$ return $\infty$
2. Return $f(x) + \mathfrak{m}_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}} \cong \mathsf{F}$.

We will later call $\mathcal{O}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}$ the residue field.

# Places

Indeed, if say $\mathfrak{p} = 5$ and $f(x) \in \mathrm{F}[x]$ then we can divide with residue ($\mathrm{F}[x]$ is a Euclidean domain) to obtain

$$f(x) = (x-5)q(x) + r(x),$$

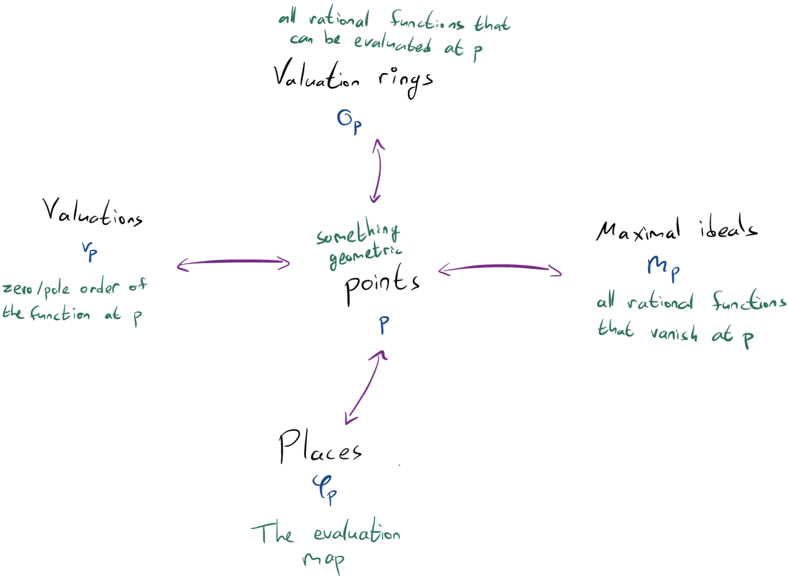
where $\deg r(x) < \deg(x-5) = 1$ and so $r(x) \in \mathrm{F}$. Thus, in $\mathcal{O}_5/\mathfrak{m}_5$,

$$f(x) + \langle x-5 \rangle = f(5) + \langle x-5 \rangle,$$

and so the unique representative in the coset $f(x) + \langle x-5 \rangle$ that belongs to F is $f(5)$.

We will later define a place to abstract such an evaluation function, also taking into account impossible evaluations (such as evaluating $\frac{1}{x-5}$ at 5.)

# To summarize



# Overview

# Curves and general fields

The suggested abstractions work great even when considering curves (not just the line F) and even over fields that are not algebraically closed.

Over finite fields (which are never algebraically closed) we have maximal ideals that do not correspond to points. E.g., $\langle x^2 + x + 1 \rangle$ is a maximal ideal in $\mathbb{F}_2[x]$.

However, the relation to points does not break entirely. Indeed, over $\mathbb{F}_4$, $x^2 + x + 1$ splits and so $\langle x^2 + x + 1 \rangle$ corresponds to a pair of Galois conjugates over $\mathbb{F}_2$.

# Curves and general fields

For curves the examples we did break down due to lack of unique factorization. Our abstraction will make use of unique factorization of ideals.

This is very much related to Kummer's erroneous attempt at proving Fermat's Last Theorem (1847). If $p$ is a prime and

$$z^p = x^p + y^p$$

then, in $\mathbb{Z}[\zeta]$, $\zeta$ being a $p$-th root of unity, we have

$$z^p = \prod_{j=0}^{p-1} x + \zeta^j y.$$

Implicitly assuming unique factorization, Kummer gave a proof by arguing about these two factorizations.

# Curves and general fields

While we will not explicitly refer to these notions from commutative algebra in the course, we remark that Dedekind domains have a unique factorization of ideals.

A domain $R$ is a Dedekind domain if it satisfies

- $R$ is Noetherian (every ideal is finitely generated).
- $R$ has Krull dimension one (every nonzero prime ideal is maximal).
- $R$ is integrally closed.

Geometrically, considering an algebraically closed field F, a curve $Z_f$ is nonsingular $\iff$ the corresponding domain

$$C_f = \mathsf{F}[x, y] \Big/ \langle f(x, y) \rangle$$

is Dedekind.