The Rate-Immediacy Barrier in Explicit Tree Code Constructions

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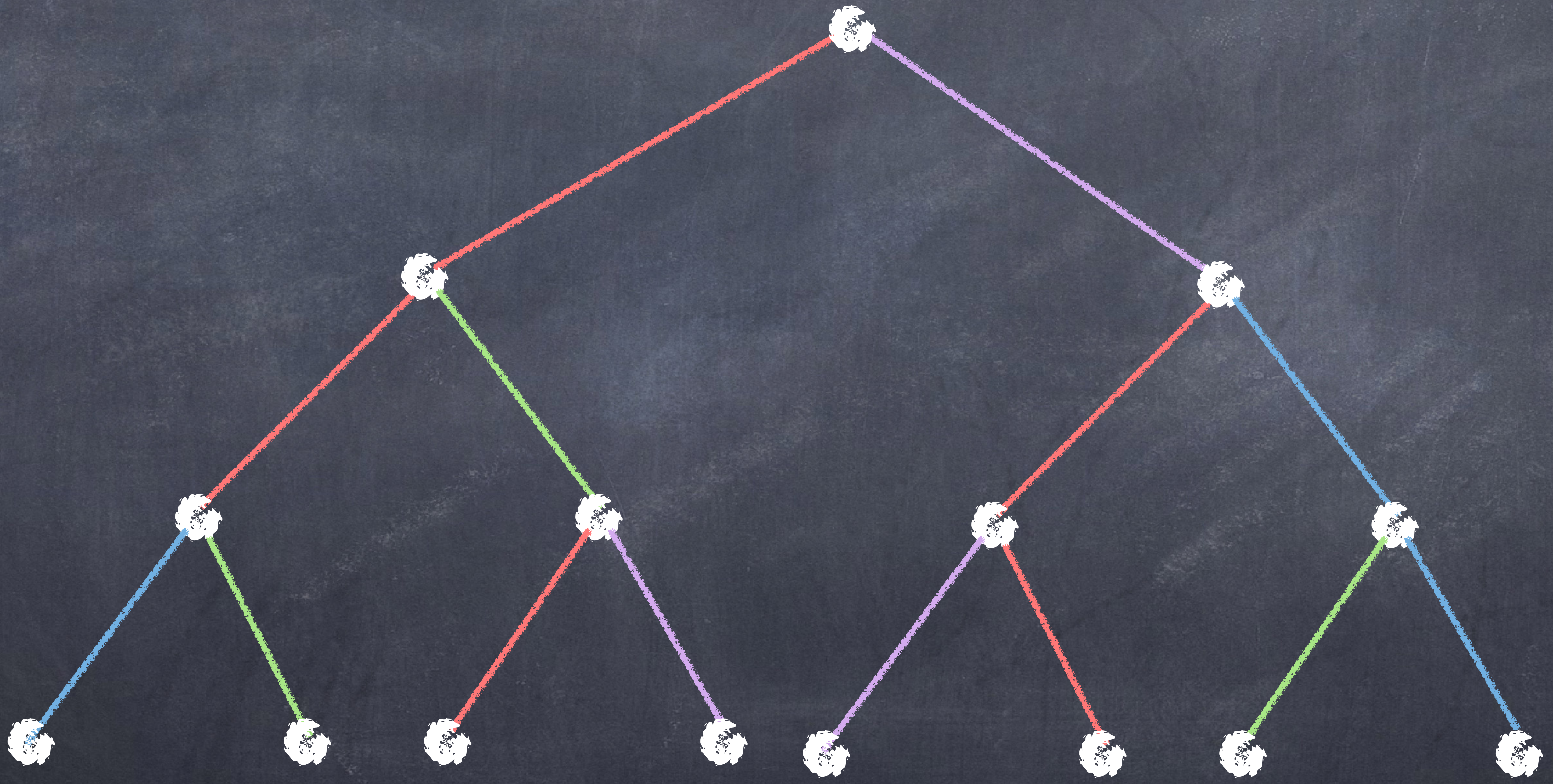
Tree Codes

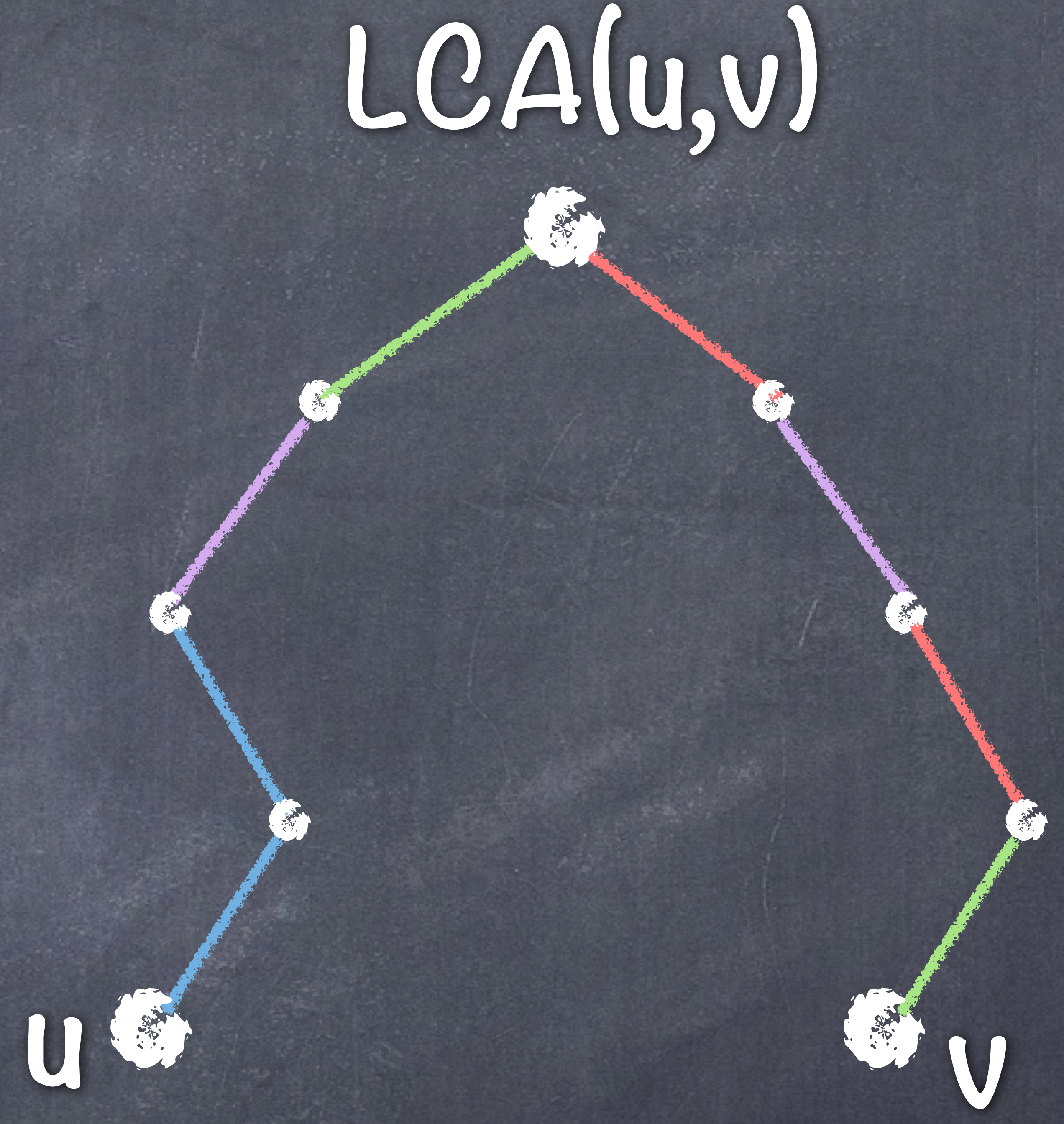
A tree code is an edge-coloring of the infinite complete rooted binary tree satisfying a distance condition.



Tree Codes

The distance δ of the tree code is given by

$$\operatorname*{Inf}_{u \neq v \text{ of equal depth}} \text{fraction of color disagreements starting from u and v's least common ancestor}$$

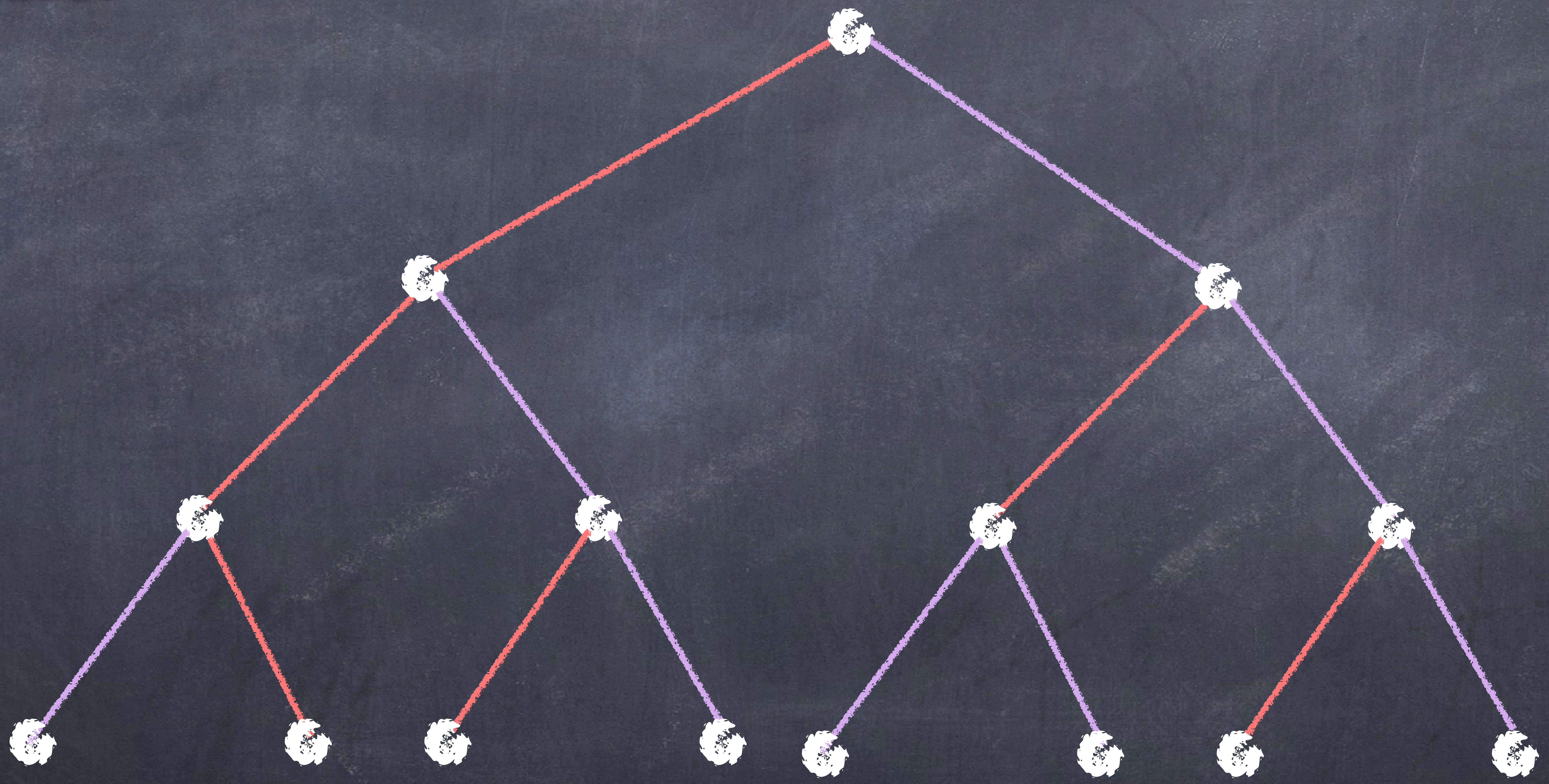


The rate is $\rho = (\log_2 c)^{-1}$, where c is the number of colors.

What is the minimum number of colors needed?

Claim. No 2-color tree code has positive distance δ.

Proof by picture.



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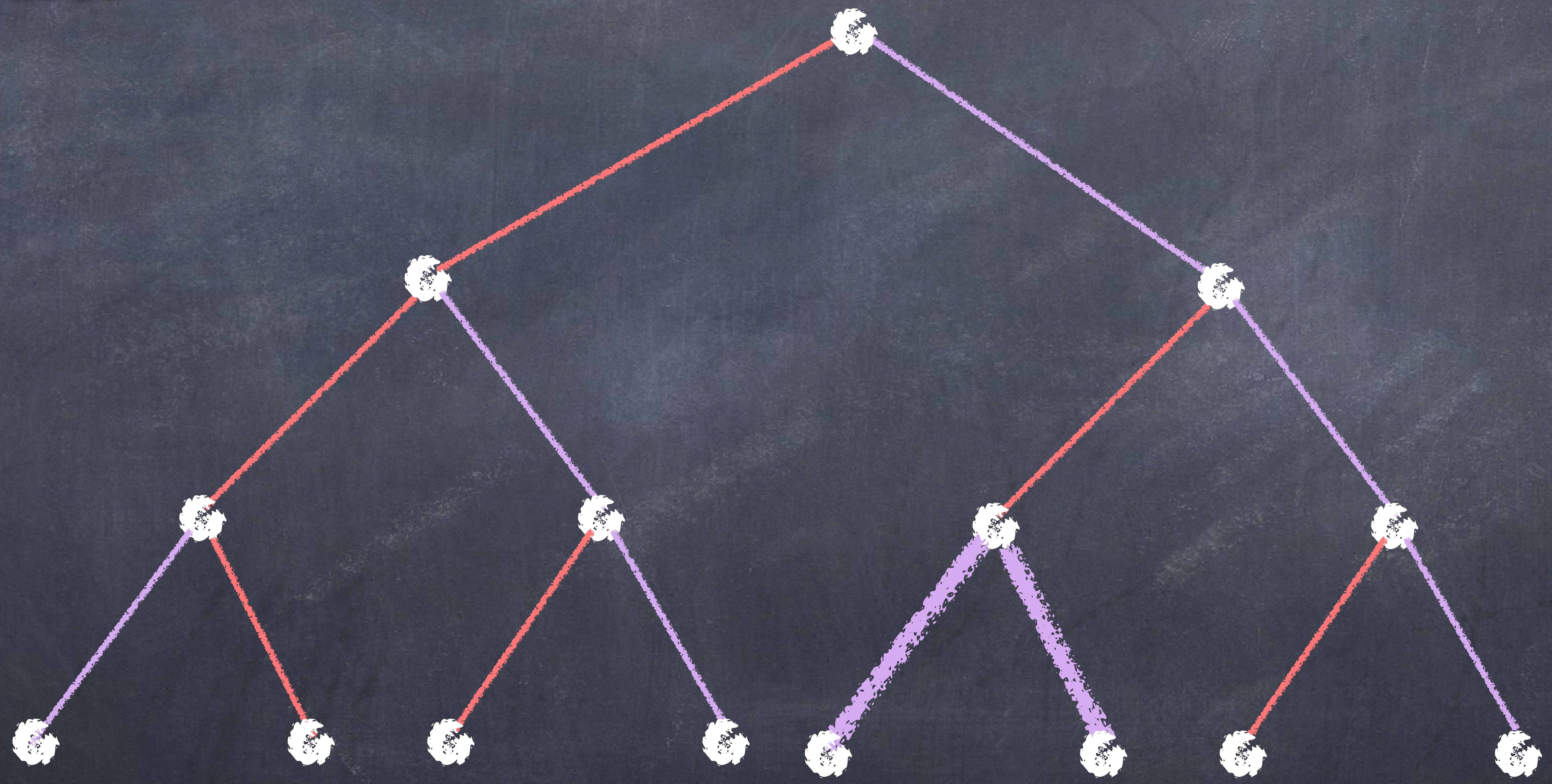

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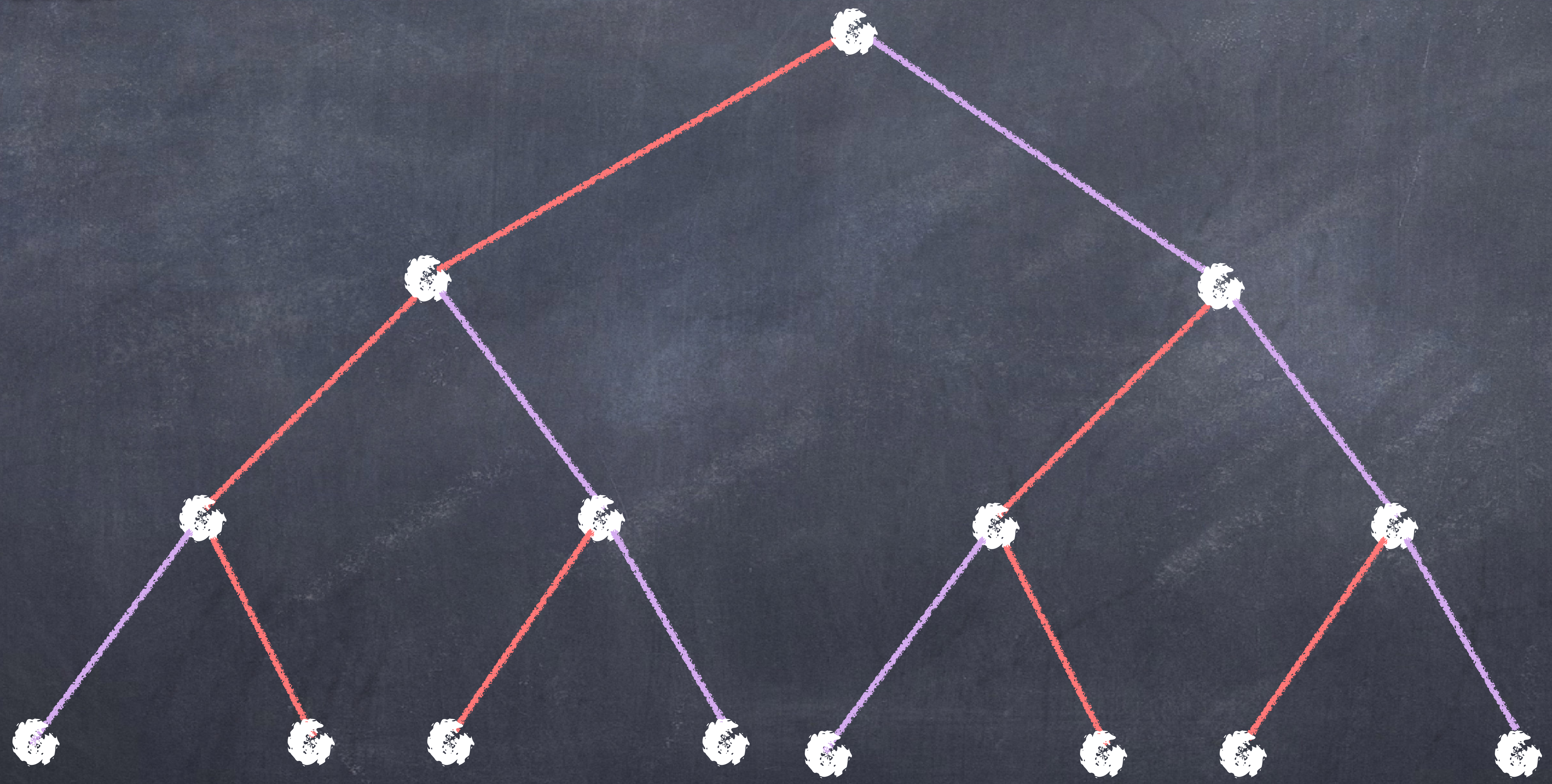

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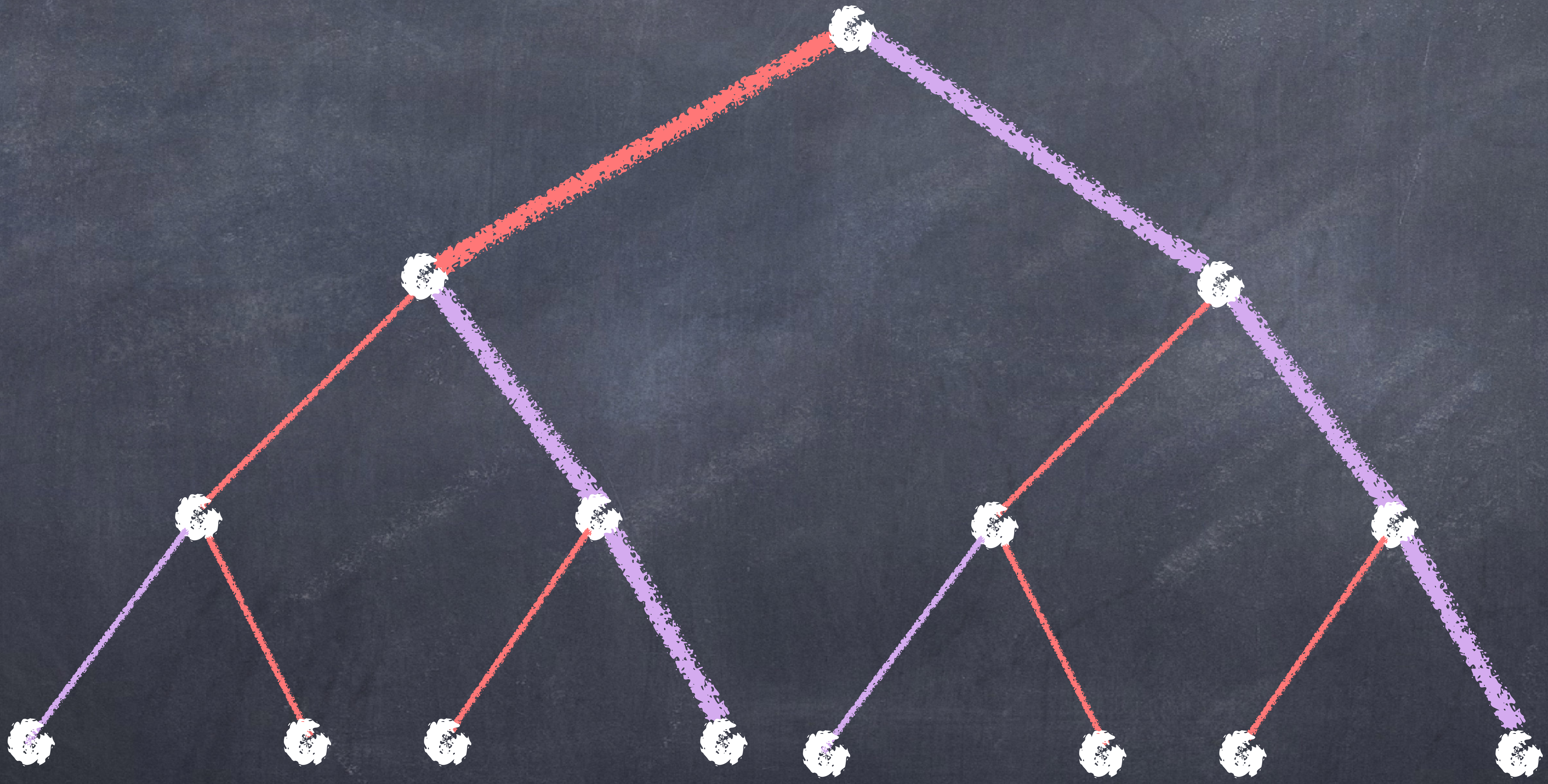

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Claim. No 3-color tree code has positive distance δ.

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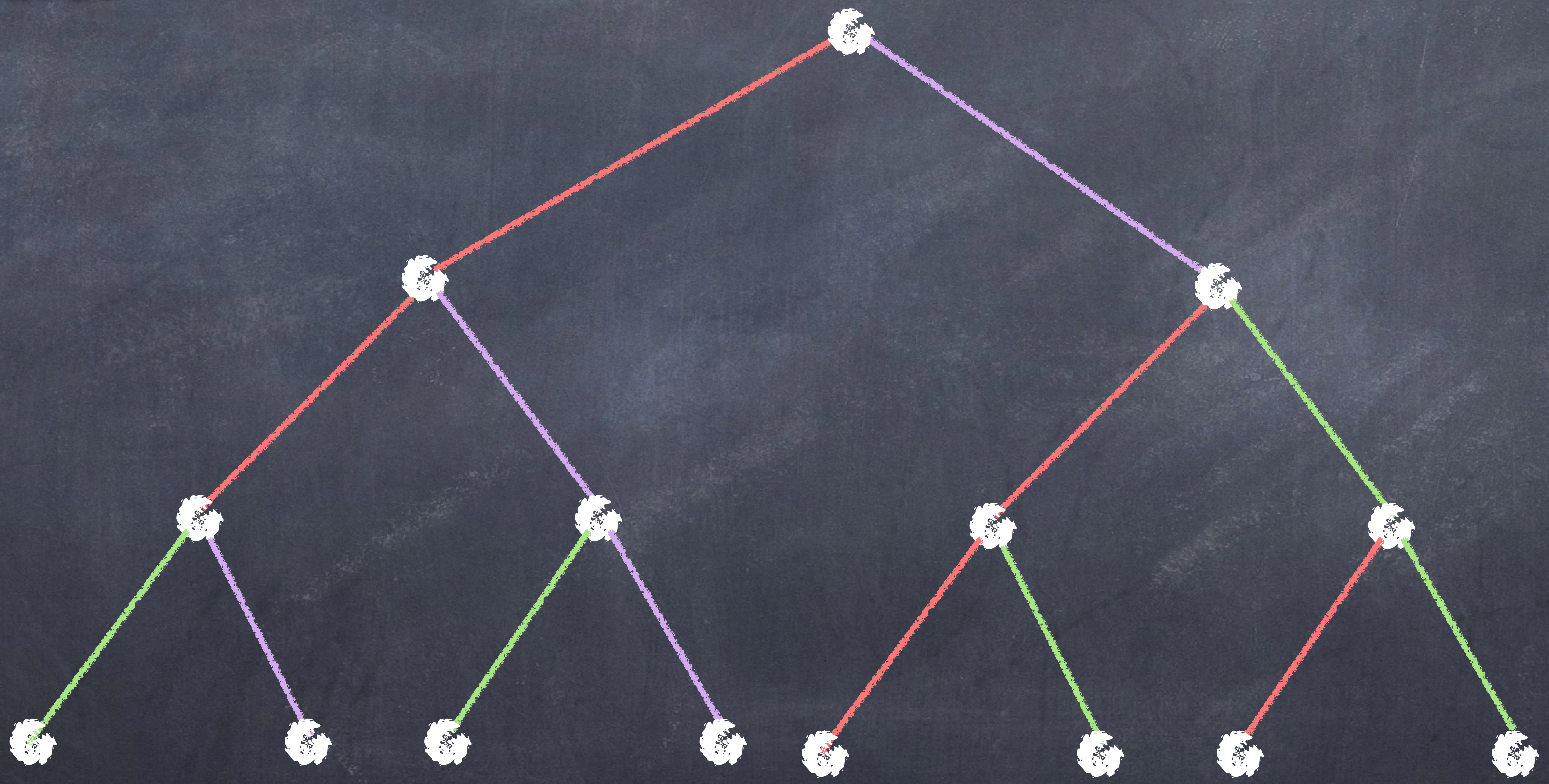

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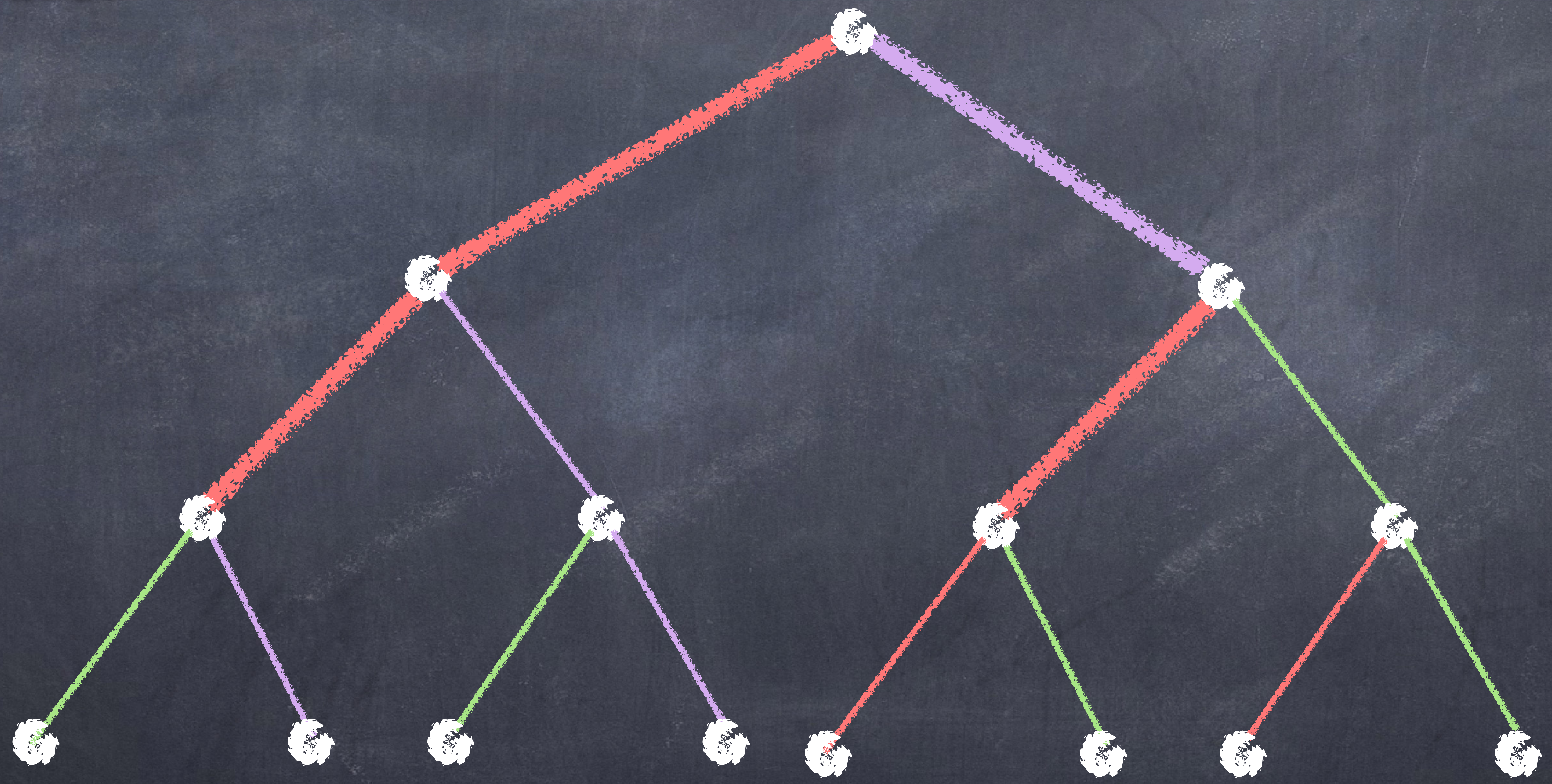

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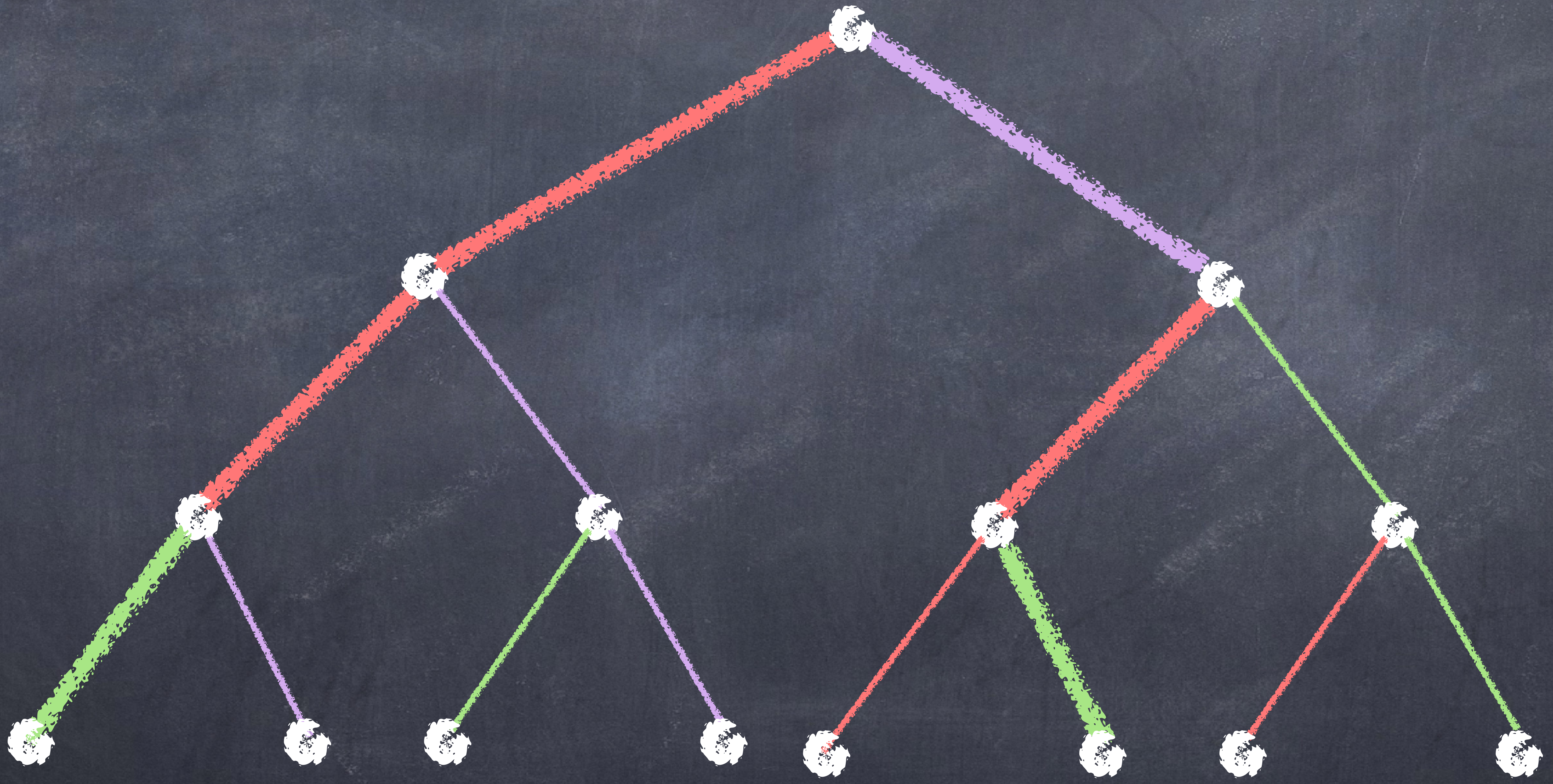

What is the minimum number of colors needed?

Theorem. There exists a 4-color tree code with distance $\delta > 0.136$.

The proof is based on a variant of Schulman's probabilistic argument [Schulman STOC 1993; C-Samocha CCC 2020].

Explicit Constructions

Explicit constructions

	Number of colors at depth n
[Evans-Klugerman-Schulman 1994]	$\mathrm{poly}(n)$
[C-Haeupler-Schulman STOC 2018]	$\mathrm{poly}(\log n)$
[Narayanan-Weidner SODA 2020]	$\mathrm{poly}(\log n)$
[BenYaacov-C-Yankovitz STOC 2022]	$(\log n)^{O(\sqrt{\delta})}$
[Yankovitz 2025]	$\mathrm{poly}(\log n)$

Explicit constructions: vanishing distance

	distance δ
[Gelles-Haeupler-Kol-RonZewi-Wigderson SODA 2016]	$(\log n)^{-1}$
[BenYaacov-C-Yankovitz STOC 2022]	$(\log \log n)^{-2}$

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Conditional asymptotically good constructions

[Moore-Schulman ITCS 2014]

[BenYaacov-C-Narayanan RANDOM 2021]

Motivating meta-questions

* All known unconditional explicit constructions make increasingly sophisticated use of block error-correcting codes.
* How far can this approach take us? Are new ideas needed?
* Despite the name, are tree codes fundamentally different from block codes?

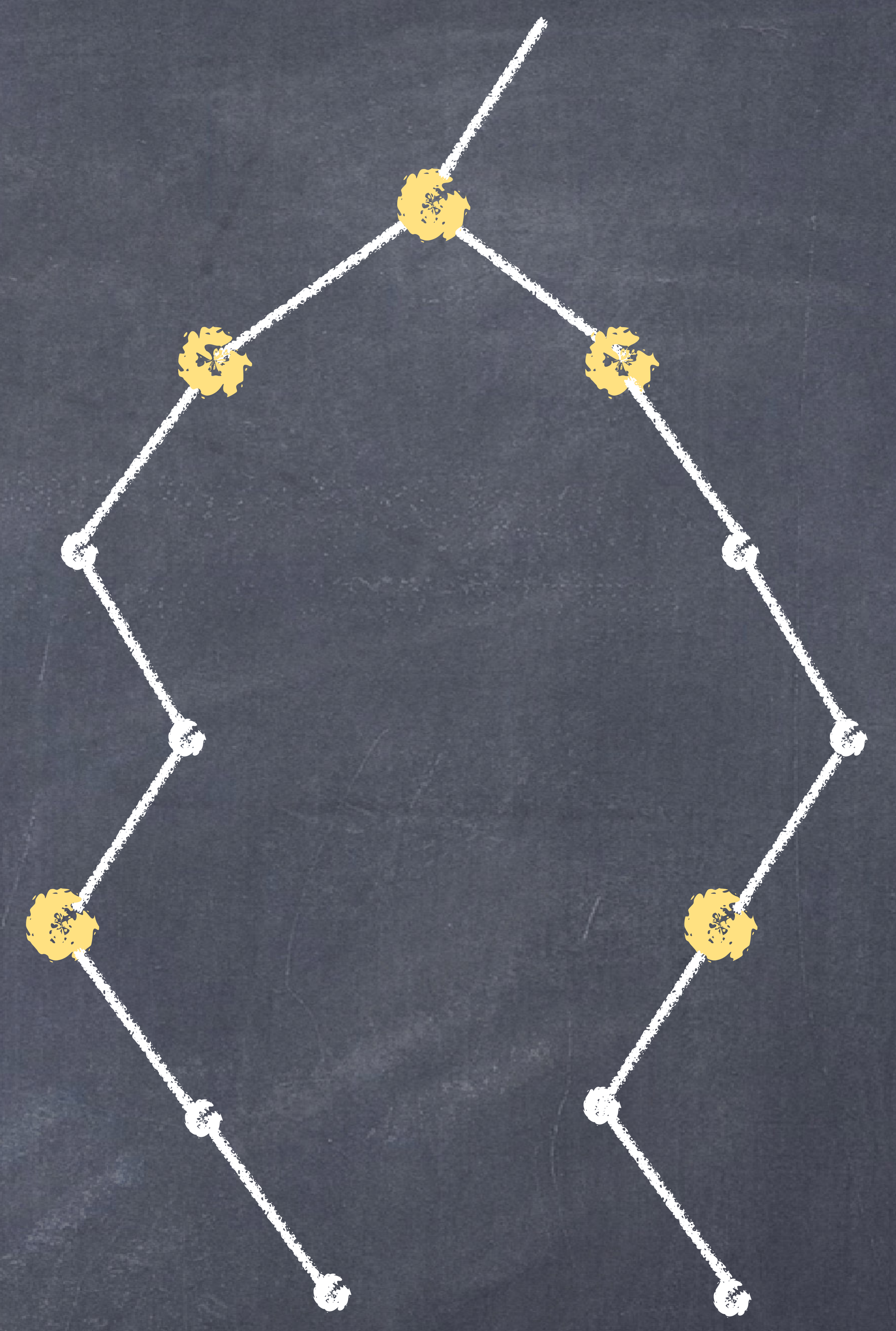
A Barrier for Current Techniques: Immediacy

Immediacy codes

The coloring is said to have immediacy

$$\mathrm{Imm} : \mathbb{N} \to \mathbb{N}$$

if for every input disagreement $x_t \neq y_t$ and every "scale" k, the output colors disagree on at least a δ-fraction of the positions $t, t+1, \ldots, t+\mathrm{Imm}(k)$.



Immediacy codes

* A standard tree code requires disagreements to accumulate only from the first input disagreement.
* Codes with immediacy support "cold-start decoding."
* This is a simplified informal definition; the formal definition uses laminar set systems.

Immediacy codes

Informally, the immediacy of the EKS construction corresponds to

$$\mathrm{Imm}_{\mathsf{EKS}}(x) = \exp(x)$$

while that of the CHS construction corresponds to

$$\mathrm{Imm}_{\mathsf{CHS}}(x) = \exp(\exp(x))$$

Their precise guarantees are slightly weaker and are handled by the formal definition.

Main Result

Theorem (informal). A code with constant distance $\delta > 0$ and immediacy function Imm has rate

$$\rho = O\left(\frac{1}{\mathrm{Imm}^{-1}(n)}\right)$$

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$$\mathrm{Imm}_{\mathrm{EKS}}(x) = \exp(x) \quad \Longrightarrow \quad \rho_{\mathrm{EKS}} = \Theta\left(\frac{1}{\log n}\right)$$

$$\mathrm{Imm}_{\mathrm{CHS}}(x) = \exp(\exp(x)) \quad \Longrightarrow \quad \rho_{\mathrm{CHS}} = \Theta\left(\frac{1}{\log\log n}\right)$$

A Toy Version of the Proof

Toy example

Theorem. A code having immediacy $\mathrm{Imm}(x) = \exp(x)$ for constant distance $\delta > 0$ has rate

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Proof sketch. First note that at the cost of halving the distance,

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We may also make the code systematic by adding one output bit per symbol, replacing c by $2c$.

Notation.

- Let X be a random variable uniformly distributed over n-bit strings.
- Partition X into its left and right halves, $X = X_L \circ X_R$.
- Let $Y = T(X) \in [c]^n$ be the corresponding color sequence, and write $Y = Y_L \circ Y_R$.

Claim.

$$I(Y_L; Y_R) = \Omega(\delta n)$$

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Let S be the suffix of L of length $\delta n/3$.

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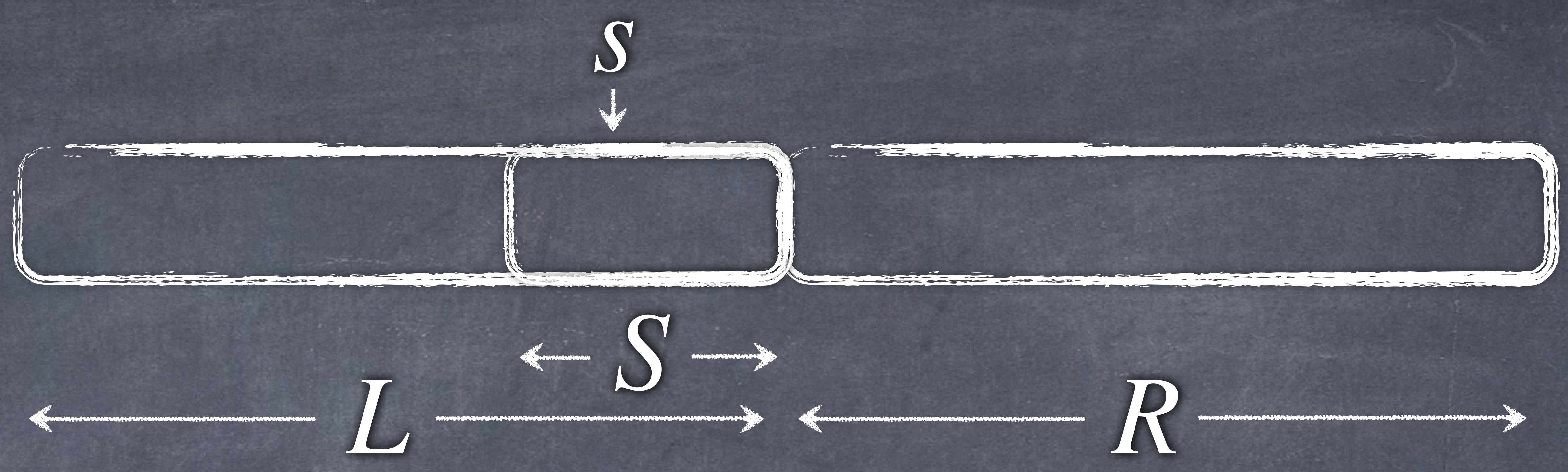

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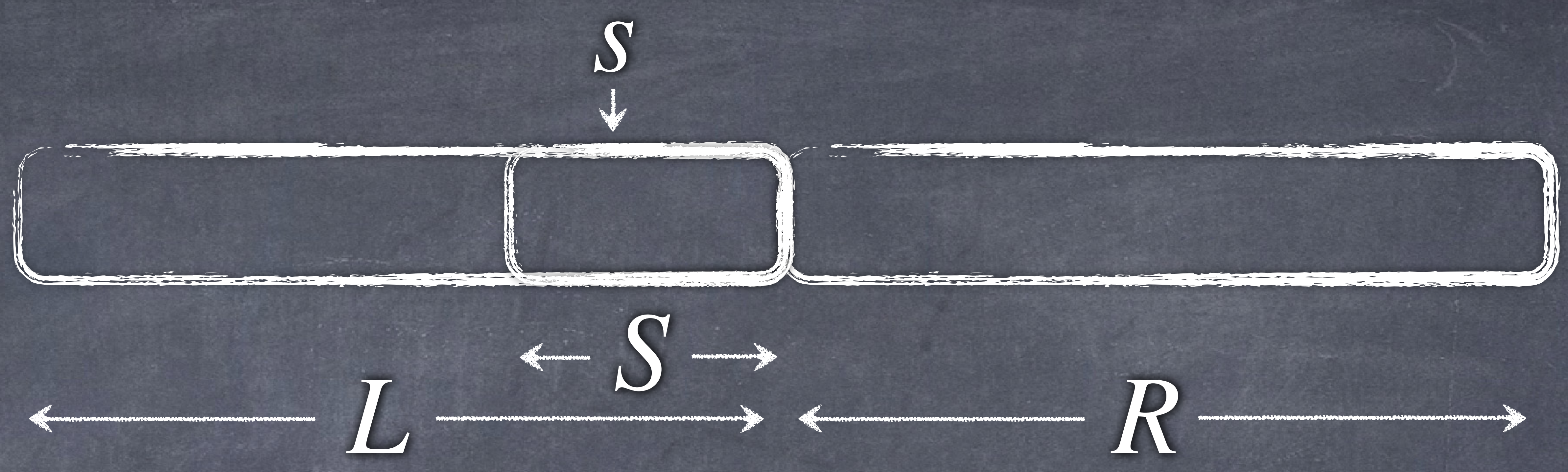
Let S be the suffix of L of length $\delta n/3$.

Suppose, toward a contradiction, that $\exists x, x' \in \{0,1\}^n$ such that

$$\exists s \in S \qquad x_s \neq x'_s \qquad T(x)_R = T(x')_R$$

Claim. $I(Y_L; Y_R) = \Omega(\delta n)$



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By immediacy,

$$\operatorname{dist}\left(T(x)_{S\cup R}, T(x')_{S\cup R}\right) \geq \frac{\delta n}{2}$$

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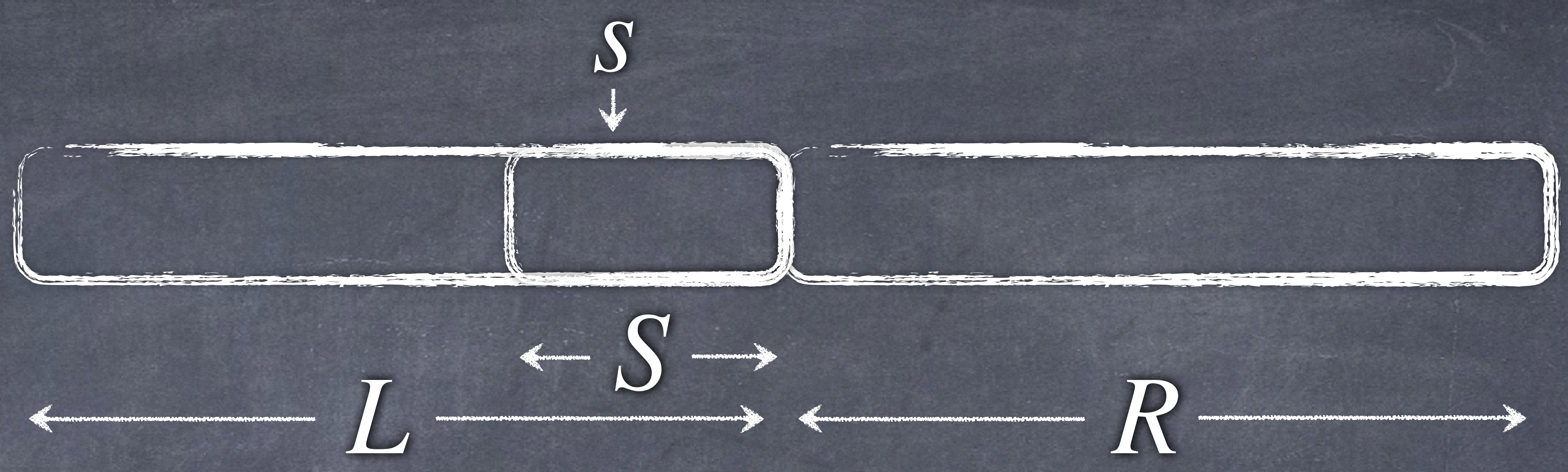

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$$\frac{\delta n}{3} = |S| \geq \operatorname{dist}\left(T(x)_S, T(x')_S\right) = \operatorname{dist}\left(T(x)_{S\cup R}, T(x')_{S\cup R}\right) \geq \frac{\delta n}{2}$$

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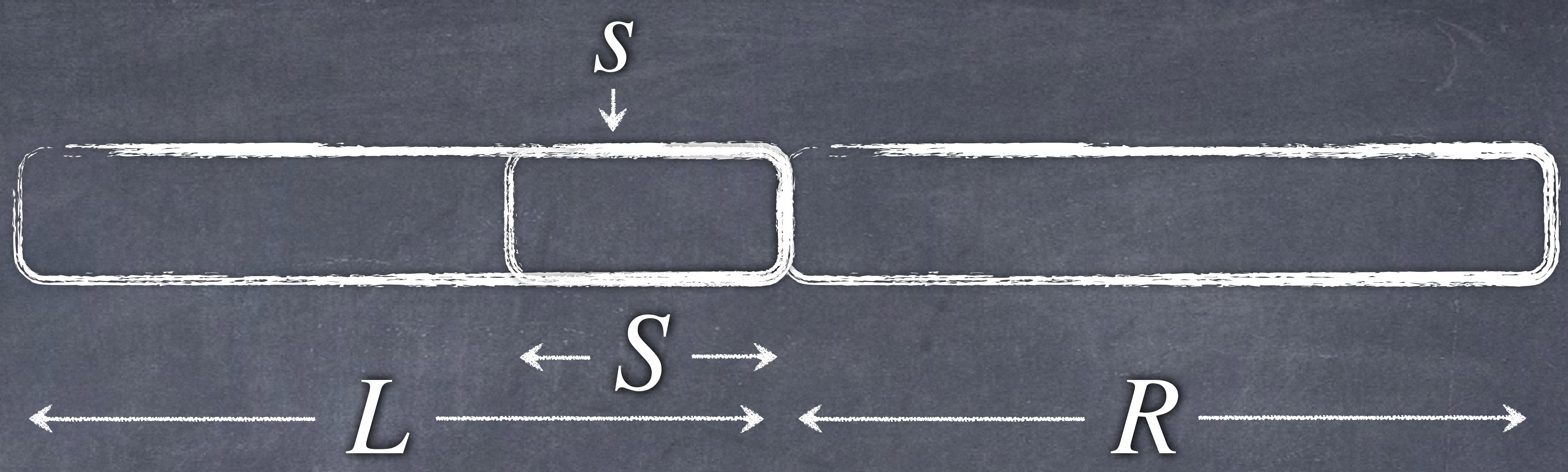

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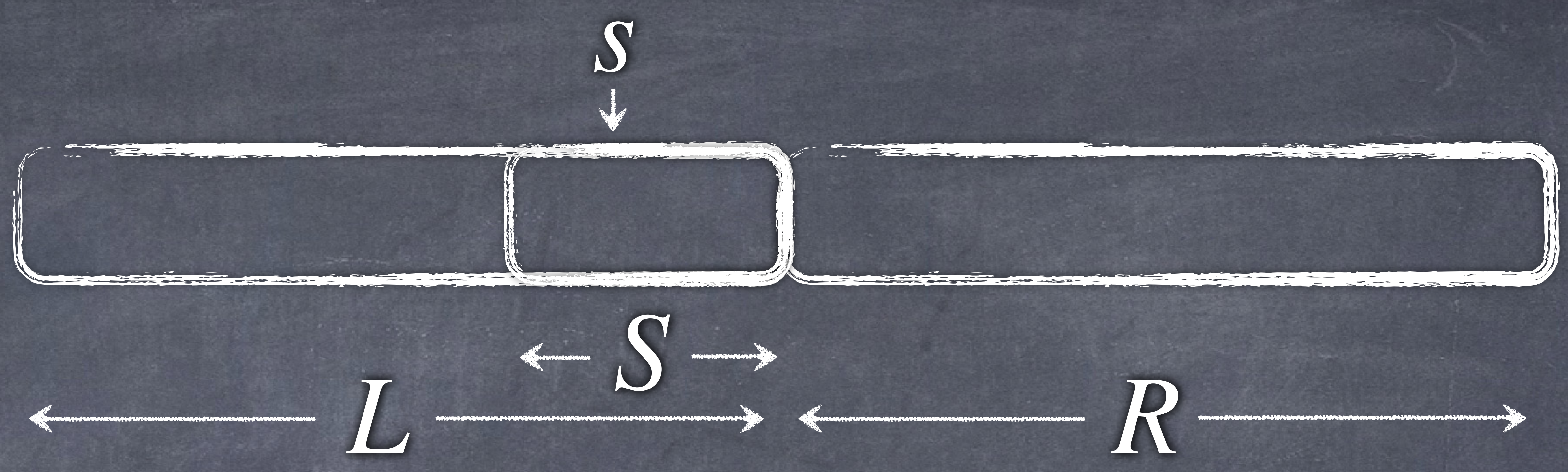

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That is, $Y_R = T(X)_R$ uniquely determines X_S .

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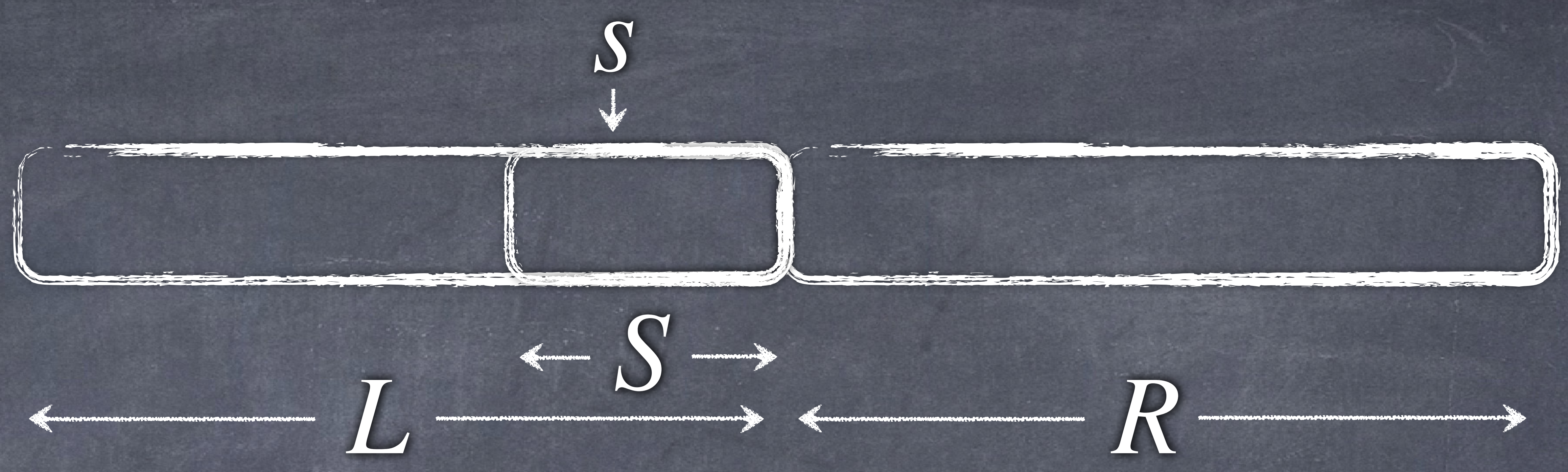

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Because the code is systematic, Y_L also determines X_S. Therefore, by data processing

$$I(Y_L; Y_R) \geq I(X_S; Y_R) = H(X_S) = \frac{\delta n}{3}$$

Claim. $I(Y_L; Y_R) = \Omega(\delta n)$

Proof of Theorem.

$$H(Y) = H(Y_L, Y_R) = H(Y_L) + H(Y_R) - I(Y_L; Y_R)$$

$$\leq H(Y_L) + H(Y_R) - \Omega(\delta n)$$

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Applying the same argument recursively to every dyadic block at every scale gives

$$H(Y) \leq \sum_{i=1}^{n} H(Y_i) - \Omega(\delta n \cdot \log n)$$

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and so, for constant distance,

$$\log_2 c = \Omega(\log n) \quad \Longrightarrow \quad \rho = O\left(\frac{1}{\log n}\right)$$

Summary

* All known unconditional constructions based on recursive combinations of block codes exhibit immediacy.

* Immediacy forces a rate loss, and the resulting upper bounds essentially match the known rates of these constructions.

* Thus, substantial progress appears to require ideas beyond recursive combinations of block codes.

Thanks!