The Ramification and Residual Indices

Unit 7

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So far

A valuation $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ induces a valuation ring

$$\mathcal{O} = \{a \in \mathsf{F} \mid v(a) \geq 0\}$$

with a unique maximal ideal

$$\mathfrak{m} = \{a \in \mathsf{F} \mid v(a) > 0\},$$

and a place

$$\varphi : \mathsf{F} \to \left(\mathcal{O} \middle/ \mathfrak{m}\right) \cup \{\infty\}$$

that extends the projection map $\mathcal{O} \mapsto \mathcal{O}/\mathfrak{m}$.

Overview

1. Valuations and friends in field extensions
2. A tale of two indices
3. Valuations in finite extensions of the rational function field

Valuations and friends in field extensions

Let E be a subfield of F and $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ a valuation with corresponding $\mathcal{O}, \mathfrak{m}, \varphi$. Observe that

$$v|_{\mathsf{E}} : \mathsf{E} \to \Gamma \cup \{\infty\}$$

is a valuation of E (additivity and triangle inequality still hold.) Further, the corresponding valuation ring is

$$\mathcal{O}_{\mathsf{E}} = \mathcal{O} \cap \mathsf{E}.$$

Indeed,

$$\begin{aligned} \mathcal{O}_{\mathsf{E}} &= \{a \in \mathsf{E} \mid v|_{\mathsf{E}}(a) \geq 0\} \\ &= \{a \in \mathsf{E} \mid v(a) \geq 0\} \\ &= \mathcal{O} \cap \mathsf{E}. \end{aligned}$$

The maximal ideal of $\mathcal{O}_{\mathsf{E}}$ is $\mathfrak{m}_{\mathsf{E}} = \mathfrak{m} \cap \mathsf{E}$ as

$$\begin{aligned} \mathfrak{m}_{\mathsf{E}} &= \{a \in \mathsf{E} \mid v|_{\mathsf{E}}(a) > 0\} \\ &= \{a \in \mathsf{E} \mid v(a) > 0\} \\ &= \mathfrak{m} \cap \mathsf{E}. \end{aligned}$$

Valuations and friends in field extensions

The induced place is then given by

$$\varphi_{\mathsf{E}} : \mathsf{E} \to \left(\mathcal{O}_{\mathsf{E}} \Big/ \mathfrak{m}_{\mathsf{E}}\right) \cup \{\infty\}.$$

We observe that

$$\mathcal{O}_{\mathsf{E}} \Big/ \mathfrak{m}_{\mathsf{E}} \hookrightarrow \mathcal{O} \Big/ \mathfrak{m}$$

via the map $a + \mathfrak{m}_{\mathsf{E}} \mapsto a + \mathfrak{m}$.

Note that this map is well-defined. Indeed, if $a + \mathfrak{m}_{\mathsf{E}} = b + \mathfrak{m}_{\mathsf{E}}$ then $a - b \in \mathfrak{m}_{\mathsf{E}} \subseteq \mathfrak{m}$, and so $a + \mathfrak{m} = b + \mathfrak{m}$.

To see that this is an embedding, take $a + \mathfrak{m}_{\mathsf{E}}$ that is mapped to $\mathfrak{m}$. Then, $a \in \mathfrak{m}$. But we also have that $a \in \mathcal{O}_{\mathsf{E}} \subseteq \mathsf{E}$ and so

$$a \in \mathfrak{m} \cap \mathsf{E} = \mathfrak{m}_{\mathsf{E}}.$$

To summarize, the residue field of $v|_{\mathsf{E}}$ is a subfield (up to isomorphism) of the residue field of v.

Overview

A tale of two indices

Definition 1 (Residual index)

Let $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ be a valuation, and E a subfield of F. The degree

$$\left[\mathcal{O}/\mathfrak{m} : \mathcal{O}_{\mathsf{E}}/\mathfrak{m}_{\mathsf{E}}\right]$$

is called the residual index of v.

Definition 2 (Ramification index)

Let $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ be a valuation, and E a subfield of F. Note that $v(\mathsf{E}^\times)$ is a subgroup of $v(\mathsf{F}^\times)$. The index

$$(v(\mathsf{F}^\times) : v(\mathsf{E}^\times))$$

is called the ramification index of v.

A tale of two indices

Proposition 3

Let $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ be a valuation, and E a subfield of F. Then,

$$\left[\mathcal{O}/\mathfrak{m} : \mathcal{O}_{\mathsf{E}}/\mathfrak{m}_{\mathsf{E}}\right] \cdot (v(\mathsf{F}^\times) : v(\mathsf{E}^\times)) \leq [\mathsf{F} : \mathsf{E}].$$

In particular, in a finite extension F/E, both indices are finite.

Proof.

In the recitation. □

Overview

A little group-theoretic claim

Claim 4

Let $\Delta \leq \Gamma$ be ordered groups with $(\Gamma : \Delta) < \infty$. Then,

$$\begin{aligned} \Delta \cong \mathbb{Z} &\implies \Gamma \cong \mathbb{Z}, \\ \Delta = 0 &\implies \Gamma = 0. \end{aligned}$$

Proof.

Let $(\Gamma : \Delta) = n$. We proved that $\varphi_n : \Gamma \to \Gamma$ that maps $\gamma \mapsto n\gamma$ is an order-preserving monomorphism. Take $\gamma \in \Gamma$. In Γ/Δ, $\gamma + \Delta$ has order dividing n, and so

$$\begin{aligned} n(\gamma + \Delta) = n\gamma + \Delta = \Delta &\implies \\ \varphi_n(\gamma) = n\gamma \in \Delta &\implies \quad \Gamma \cong \varphi_n(\Gamma) \leq \Delta. \end{aligned}$$

Thus, $\Delta = 0 \implies \Gamma = 0$.

Now, if $\Delta \cong \mathbb{Z}$ then either $\Gamma \cong \mathbb{Z}$ or $\Gamma = 0$. The latter case cannot hold as $\mathbb{Z} \cong \Delta \leq \Gamma$. □

Valuations in finite extensions of the rational function field

Recall that a valuation $v : \mathsf{F} \to \Gamma \cup \{\infty\}$ is trivial if $v(\mathsf{F}^\times) = 0$.

Corollary 5

Let F be a finite extension of $\mathsf{E} = \mathsf{K}(t)$. Then, every non-trivial valuation v of F that is trivial on K is discrete.

Proof.

By Proposition 3,

$$(v(\mathsf{F}^\times) : v(\mathsf{E}^\times)) \leq [\mathsf{F} : \mathsf{E}] < \infty.$$

By Claim 4 and since $v(\mathsf{F}^\times) \neq 0$ we have $v(\mathsf{E}^\times) \neq 0$.

In the recitations you will characterize all valuations of $\mathsf{E} = \mathsf{K}(t)$, and in particular show that they are discrete. Thus, $v_{|\mathsf{E}}$ is discrete, and so $v(\mathsf{E}^\times) \cong \mathbb{Z}$.

Applying Claim 4 again implies that v is also discrete. □