Free Probability and Ramanujan Graphs - HW #5

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Exercises with * will be graded. Submission in singles or pairs. Contact Gal for questions and clarifications.

1. * Let μ be a probability measure and let $r \in \mathbb{R}$. We denote by $S_r(\mu)$ the shift of μ by r, that is, for every measurable set $A \subseteq \mathbb{R}$:

$$S_r(\mu)(A) = \mu(A - r),$$

   where $A - r = \{x - r : x \in A\}$. Show that $\mu \boxplus \delta_r = S_r(\mu)$.

2. * Let $d \in \mathbb{N}$. We donte by $\mu^{\boxplus d}$ the convolution of μ with itself d times (e.g. $\mu^{\boxplus 2} = \mu \boxplus \mu$). Derive the probability measure $\mu^{\boxplus d}$ where μ is the probability measure of a random variable c which is -1 with probability $\frac{2}{3}$, and 2 with probability $\frac{1}{3}$.

   Explain in your own words (informally), what is the meaning of the above calculation in the context of graph theory.

3. * Let b be a Rademacher random variable, that is, $b = \pm 1$ each with probability $\frac{1}{2}$. Prove that the free cumulants of b are given by

$$\kappa_n(b, \dots, b) = \begin{cases} (-1)^{p-1} C_{p-1}, & n = 2p, \\ 0, & n \text{ is odd.} \end{cases}$$

4. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $(\kappa_n)_{n \in \mathbb{N}}$ be its free cumulant functionals, and let $(a_i)_{i \in [m]}$ be random variables in $\mathcal{A}$, for some $m \in \mathbb{N}$. Prove that the following two statement are equivalent:

   (a) $(a_i)_{i \in [m]}$ are free.

   (b) For all $n \geq 2$ and $i(1), \dots, i(n) \in [m]$, $\kappa_n(a_{i(1)}, \dots, a_{i(n)}) = 0$ whenever $i(\cdot)$ is not constant (at least two indices are distinct).