Introduction to Algebraic-Geometric Codes **Spring 2019**

Exercise 6

Publish Date: 15 May 19 *Due Date: 30 May 19*

Exercise 6.1. (a) *Let $(A, \mathfrak{m})$ be a local ring, and let $a \in A \setminus \mathfrak{m}$. Show that for every $m \in \mathfrak{m}$, $a + m$ is a unit.*

(b) *Let $(A, \mathfrak{m})$ be a local ring. Show that A is a local PID $\iff$ A is noetherian and $\mathfrak{m}$ is principal.*

(c) *Let $(A, \mathfrak{m}), (B, \mathfrak{n})$ be local PID's with the same field of fractions, K, such that $A \subseteq B$. Show that $A = B$ or $B = K$.*

Exercise 6.2. *Let $A \subset B$ be an integral ring extension. Let $P \in Spec(A)$, $Q \in Spec(B)$ such that $P = Q \cap A$.*

(a) *Show that B/Q is integral over A/P.*

(b) *Show that B_P is integral over A_P.*

Exercise 6.3. *Let $A \subset B$ be an integral ring extension.*

(a) *Let $P_1 \subseteq P_2 \in Spec(A)$ and $Q_1 \in Spec(B)$ such that $P_1 = Q_1 \cap A$ show that there is $Q_1 \subseteq Q_2 \in Spec(B)$ such that $P_2 = Q_2 \cap A$.*

(b) *Deduce the following statement: Let $P_1 \subseteq ... \subseteq P_n \in Spec(A)$ and $Q_1 \subseteq ... \subseteq Q_m \in Spec(B)$. Assume that $m < n$ and that for every $1 \leq i \leq m$, $P_i = Q_i \cap A$. Then there are $Q_m \subseteq Q_{m+1} \subseteq ... \subset Q_n$ such that for every $1 \leq i \leq n$, $P_i = Q_i \cap A$. This is called the Going Up Theorem.*

(c) *Prove that $\dim(A) = \dim(B)$.*

Exercise 6.4. *Let $I_1, \dots I_n \triangleleft A$ be pairwise coprime ideals, then $A/(\prod_{i=1}^{n} I_i) \cong A/I_1 \times \dots \times A/I_n$. This is called the Chinese Remainders Theorem.*