Recitation 3: Circuits and Entanglement

3.1 Garbage and Measurements

We saw in the lecture the problem of entanglement to ”garbage”, which are ancilla qubits we use to calculate, but if we end up having the answer qubits entangled to them, we don't get the desired outcome. We saw that in the no-cloning theorem. Let's revisit this point:

If instead of $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ we get the state $\frac{1}{\sqrt{2}}(|0\rangle |g(0)\rangle + |1\rangle |g(1)\rangle)$ for some $|g(0)\rangle \neq |g(1)\rangle$. How can we see this is not equivalent to us having $|+\rangle$ at hand?

Answer. The most trivial operation would be measuring in the Hadamard basis – which includes the desired qubit state, so should have it as the outcome with certainty. The probability of this outcome in our case, though, is $\left\| \langle +|_1 \frac{1}{\sqrt{2}}(|0\rangle_1 |g(0)\rangle_2 + |1\rangle_1 |g(1)_2\rangle) \right\|^2 = \frac{1}{2} \left\| \frac{1}{\sqrt{2}} |g(0)\rangle + \frac{1}{\sqrt{2}} |g(1)\rangle \right\|^2$ so for any $|g(0)\rangle \neq |g(1)\rangle$ we get a probability that is strictly less than the expected 1! △

We ended here. Answers are suppressed below for now.

The garbage prevents the states to interfere in the way we wanted them. Can we use that to our advantage in some way? Let's see another demonstration of this interference-prevention:

We will look at the Hadamard transformation. It is its own inverse, so $H^2 = Id$. We'll write and calculate it on $|0\rangle$ using “Feynman's paths”. See Figure 3.1.

If we introduce a measurement in the middle we will prevent the interference that allows Hadamard to reverse itself. See Figure 3.2.

What happens if instead of a measurement we would perform a CNOT to a fresh $|0\rangle$ qubit? See Figure 3.3.

We see how the CNOT separates the worlds down the line, preventing interference between them, just like the measurement. In fact it is equivalent to the measurement as we could measure just before considering the original

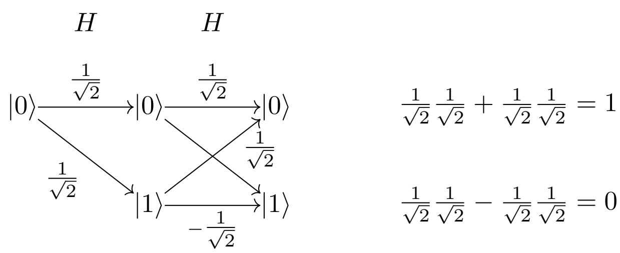

Figure 3.1: Feynman paths for $HH|0\rangle$ show the constructive interference for $|0\rangle$ and the destructive interference for $|1\rangle$.

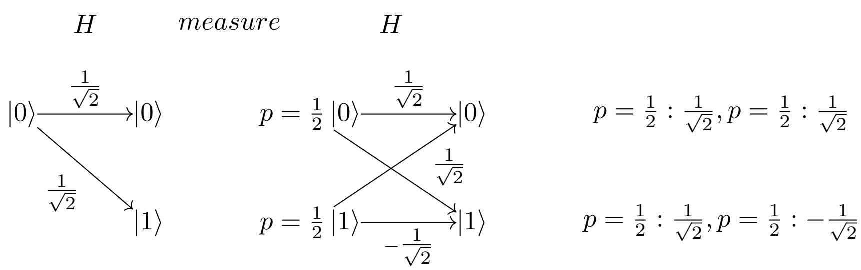

Figure 3.2: Feynman paths for $H \to measure \to H$ starting with $|0\rangle$ show the interference can no longer take place because of the measurement.

qubit. This is called the *delayed measurement principal* and we might see a more general and complete proof for it later in the course.

3.2 Circuits

1. Calculate and describe what the circuit in Figure 3.4 does. (Question 1 in the Circuits part of Ex3)

2. We said we can measure in any orthonormal basis, and even in any orthogonal decomposition of the space. But the circuit formalism normally uses measurements in the computational basis. How can we implement a measurement in another basis $\{|v_i\rangle\}_{i=0}^{d-1}$ using unitary transformations and computational-basis measurements? (question 2 in the Circuits part of Ex3)

   Let's generalise this question to the slightly more general measurement, by an example: create a circuit that prepares, with success probability $\frac{1}{2}$, the state we looked at in the last recitation: $\frac{\sqrt{2}}{\sqrt{d}}\sum_{\substack{i\in\{0,\ldots,d-1\}\\ i\equiv 0 \mod 2}}|i\rangle$, where $d := 2^n$, for n qubits, and $|i\rangle$ is the computational basis that is i in binary. Assume you can have a unitary U_f on $n+1$ qubits, for any boolean function $f : \{0,1\}^n \to \{0,1\}$ that you know how to calculate classically, such that

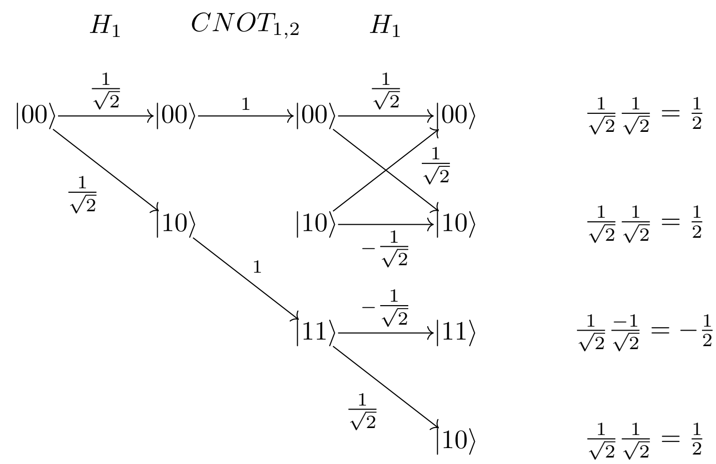

Figure 3.3: Feynman paths for $H(CNOT)H\,|00\rangle$ show the interference can no longer take place, this time because of the CNOT.

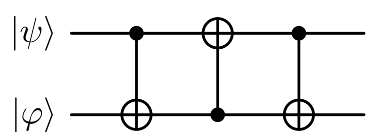

Figure 3.4: Circuit on 2 qubits. the gates are all CNOT, where the $\bullet$ is the control qubit and the $\oplus$ is the target qubit.

$$U_f\,|xy\rangle = |x(f(x) \oplus y)\rangle \text{ for all } x \in \{0,1\}^n, y \in \{0,1\}.$$