BILU – LINIAL EXPANDERS

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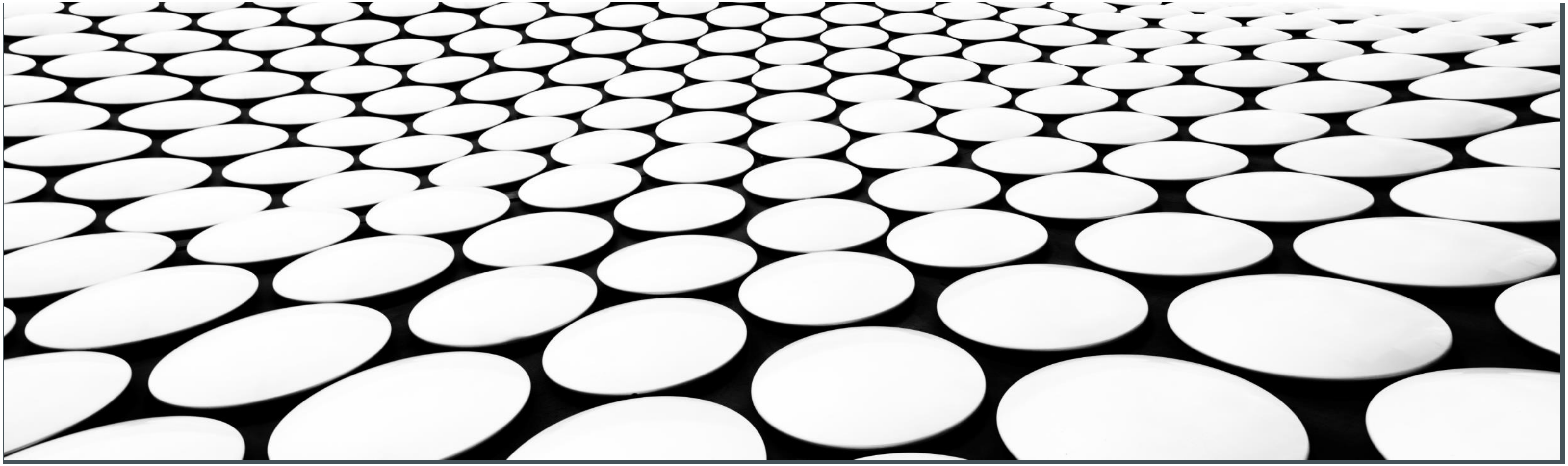
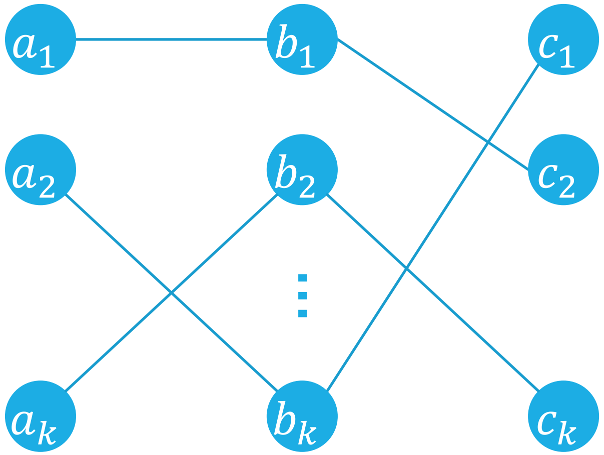


BRIEF HISTORY

- 1970s — Expanders exist
- 1988 — Ramanujan graphs construction
- 1991 — Alon Boppana Thm
- 2002 — Zig-Zag product $O(d^{\frac{3}{4}})$ expanders
- 2002 — $O(d)$ Expanders via the line graph
- 2003 — $O\left(\sqrt{d \cdot \log^3(d)}\right)$ Bilu – Linial expanders

GRAPH LIFTS

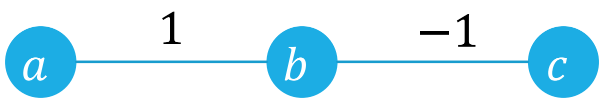
- A k-lift of a graph $G = (V, E)$ is a new graph $\hat{G} = (V', E')$
  - Each vertex v in G corresponds k vertices in $\widehat{\mathrm{G}}$, called the fiber of v
  - Each edge $e = (u, v)$ in G corresponds a perfect matching in the fibers of u and v
- If G is d regular, so is $\hat{G}$

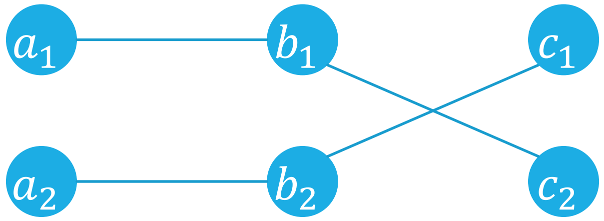


2-LIFTS

- A 2-lift of a graph
  - Each vertex v in G corresponds 2 vertices in $\hat{G}$, called the fiber of v
  - Each edge $e = (u, v)$ in G corresponds **one of two** perfect matchings in the fibers of u and v

- Description of a 2-lift can thus be done with one bit per edge
- Let 1 correspond "parallel" matching and let -1 correspond "crossing" matching

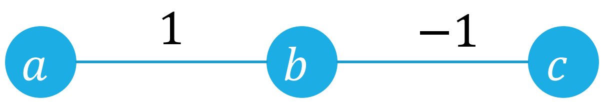


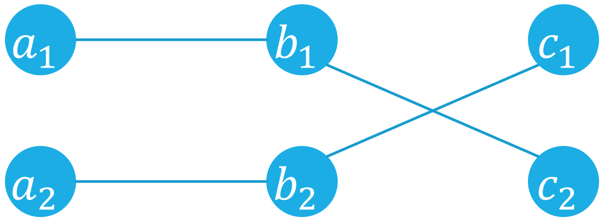


2-LIFTS

- Let $s: E \to \{\pm 1\}$ be a signing of the edges
  - $M_{u,v} = \begin{cases} 1 & u \sim v \\ 0 & else \end{cases}$
  - $[M_s]_{u,v} = \begin{cases} s((u,v)) & u \sim v \\ 0 & else \end{cases}$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$



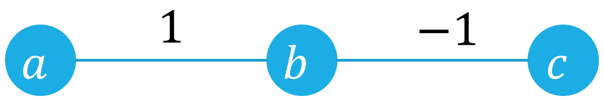


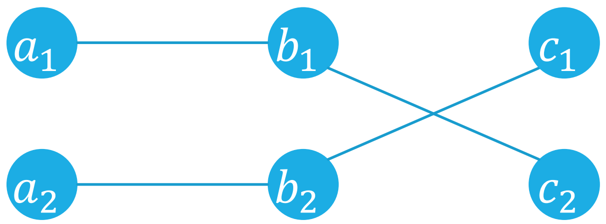
2-LIFTS LINEAR ALGEBRAIC PROPERTIES

- Denote $M_1 = \begin{cases} 1 & M_s = 1 \\ 0 & M_s \neq 1 \end{cases}$ and $M_{-1} = \begin{cases} 1 & M_s = -1 \\ 0 & M_s \neq -1 \end{cases}$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Note that $M = M_1 + M_{-1}$ and that $M_s = M_1 - M_{-1}$
- Express $M_{\hat{G}}$ using M_1 and M_{-1}
- $M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix}$





2-LIFTS EIGENVECTORS AND EIGENVALUES

- $M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix}, \quad M = M_1 + M_{-1}, \qquad M_s = M_1 - M_{-1}$
- Let v be an eigenvector of M corresponding an eigenvalue μ. Look at $\binom{v}{v}$

$$M_{\hat{G}} \cdot \begin{pmatrix} v \\ v \end{pmatrix} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix} \cdot \begin{pmatrix} v \\ v \end{pmatrix} = \begin{pmatrix} (M_1 + M_{-1})v \\ (M_1 + M_{-1})v \end{pmatrix} = \begin{pmatrix} Mv \\ Mv \end{pmatrix} = \mu \begin{pmatrix} v \\ v \end{pmatrix}$$

- Let v be an eigenvector of M_s corresponding an eigenvalue μ. Look at $\binom{v}{-v}$

$$M_{\hat{G}} \cdot \begin{pmatrix} v \\ -v \end{pmatrix} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix} \cdot \begin{pmatrix} v \\ -v \end{pmatrix} = \begin{pmatrix} (M_1 - M_{-1})v \\ (M_{-1} - M_1)v \end{pmatrix} = \begin{pmatrix} M_s v \\ -M_s v \end{pmatrix} = \mu \begin{pmatrix} v \\ -v \end{pmatrix}$$

- These are $2n$ orthogonal eigenvectors of $M_{\hat{G}}$

PLAN

- Start with a good graph G_0 and 2-lift it many times
- Each time, choose a signing s with small eigenvalues
  - How can we find such a signing?
  - Does it even exist?

GOOD SIGNINGS EXIST

- Every graph of maximal degree d has a signing with spectral radius $O\left(\sqrt{d \cdot \log^3(d)}\right)$

PROOF STEPS

- There exists a signing s such that $\frac{|u^t M_s v|}{\|u\|\|v\|} \leq 10\sqrt{d \log d}$ for all $u, v \in \{0,1\}^n$

- Let $A \in S^n(\mathbb{R})$ such that $\|A_i\|_1 \leq d$ and $|A_{i,i}| = O\left(\alpha\left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$.

  If for any two vectors $u, v \in \{0,1\}^n$ we have $\frac{u^t A v}{\|u\|\|v\|} \leq \alpha$

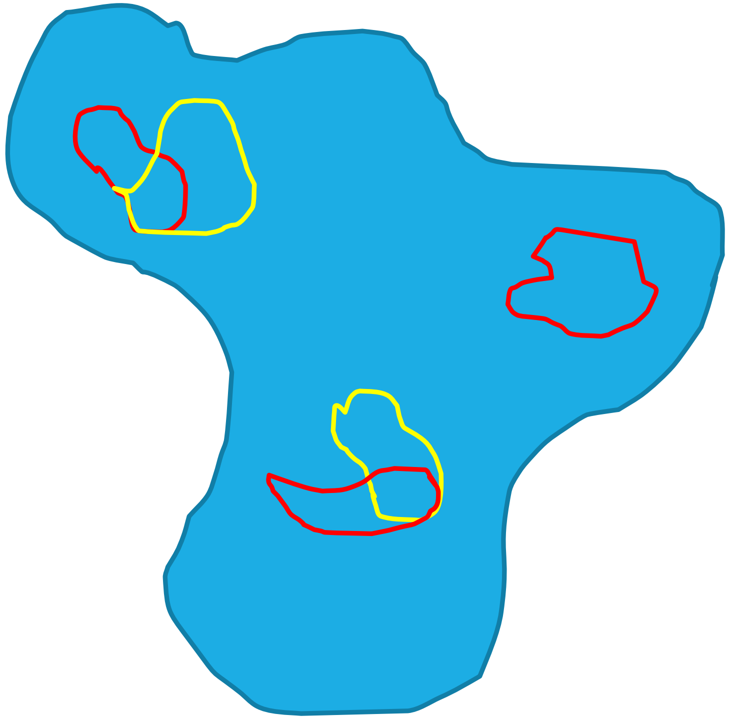
  Then the spectral radius of A is $O\left(\alpha\left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$

- Plug $\alpha = 10\sqrt{d \cdot \log(d)}$ to obtain the desired result

GOOD SIGNINGS EXIST

- There exists a signing s such that $\frac{|u^t M_s v|}{\|u\|\|v\|} \leq 10\sqrt{d \log d}$ for all $u, v \in \{0,1\}^n$

- The vertices are $\mathrm{V} = \{1 \dots n\}$
- Denote $S(v) = \{i \in [n] | v_i \neq 0\}$ the support of v
- It suffices to prove for u, v such that $S(u, v)$ is a connected component in G
  - Denote $S_1 \dots S_k$ the connected components of $S(u, v)$ and decompose $u = \sum u_i, v = \sum v_i$
  - For every $i \neq j$ we have $u_i^t M_s v_j = 0$



$$|u^t M_s v| \leq \sum |u_i^t M_s v_i| \leq \sum_{i=1}^{k} 10\sqrt{d \log(d)}\, \|u_i\|\|v_i\| \leq 10\sqrt{d \log(d)}\, \|u\|\|v\|$$

GOOD SIGNINGS EXIST

- There exists a signing s such that $\frac{|u^t M_s v|}{\|u\|\|v\|} \leq 10\sqrt{d \log d}$ for all $u, v \in \{0,1\}^n$ when $S(u,v)$ is a connected component
- Express $u^t M_s v$ as $\sum_{i \in S(v), j \in S(u)} (M_s)_{i,j}$
  - $(M_s)_{i,j}$ are independent random variables attaining values in ± 1 or ± 2 with probability $\frac{1}{2}$
  - $\mathbb{E}(M_s) = 0$
  - Summing over $e\big(S(u), S(v)\big)$ elements

- Define $B_{u,v}$ the bad event $|u^t M_s v| > 10\sqrt{d \log d}\ \|u\|\|v\|$
- Hoeffding's Inequality

$$\mathbb{P}[|X - \mu| > t] \leq 2 \exp\left(\frac{2t^2}{\sum_{i=1}^{m} (b_i - a_i)^2}\right)$$

GOOD SIGNINGS EXIST

- W.l.o.g, $|S(u)| \geq \frac{1}{2}|S(u,v)|$

$$\mathbb{P}\left[B_{u,v}\right] = \mathbb{P}\left[|v^t M_s u - 0| > 10\sqrt{d \log d}\ \|u\|\|v\|\right]$$

$$\leq 2\exp\left(-\frac{2 \cdot 100d \cdot \log(d)\ |S(u)||S(v)|}{16e\big(S(u), S(v)\big)}\right)$$

$$\leq 2\exp\left(-\frac{100d \cdot \log(d)\ |S(u)||S(v)|}{8d|S(v)|}\right)$$

$$\leq 2\exp\left(-\frac{100\log(d)\,\frac{|S(u,v)|}{2}}{8}\right)$$

$$\leq d^{-6|S(u,v)|}$$

GOOD SIGNINGS EXIST

- For each pair of vectors $u, v \in \{0,1\}^n$ there is a small probability to fail. $\mathbb{P}\left[B_{u,v}\right] \leq d^{-6|S(u,v)|}$
- How many such pairs of vectors are there?
- Union bound wouldn't suffice

LOVASZ LOCAL LEMMA

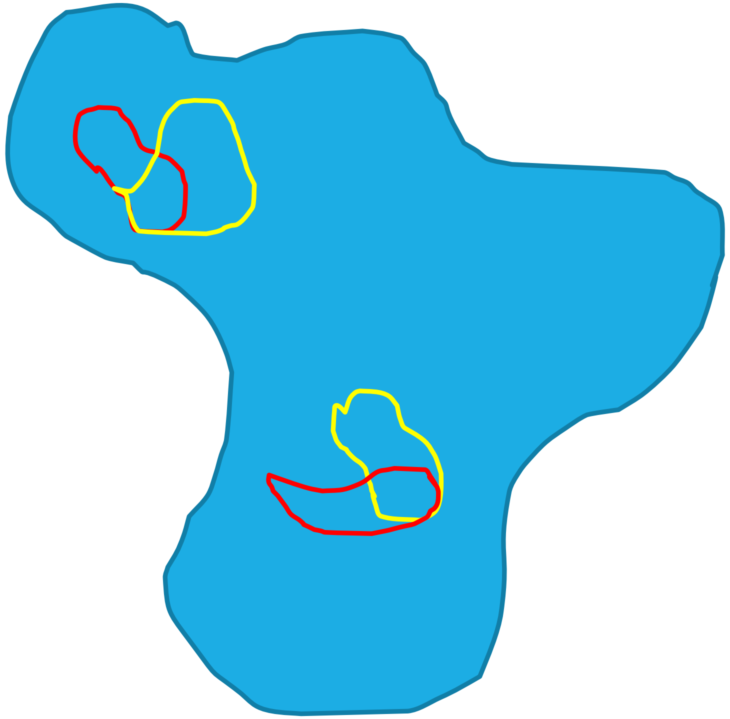
- Let H be a dependency graph
  - $A_1 \dots A_n$ are vertices corresponding bad events
  - $A_i \sim A_j$ if A_i and A_j are dependent
  - Bounded degree d
- If $\mathbb{P}[A_i] \leq \frac{1}{4d}$ then $\mathbb{P}\left[\overline{A_1} \wedge \cdots \wedge \overline{A_n}\right] > 0$

- If we assign each event A_i a quantity $x_i \in [0,1]$ such that for all i we get $\mathbb{P}[A_i] \leq x_i \prod_{A_j \sim A_i} (1 - x_j)$

$$\mathbb{P}\left[\overline{A_1} \wedge \cdots \wedge \overline{A_n}\right] > \prod_{i=1}^{n} (1 - x_i) > 0$$

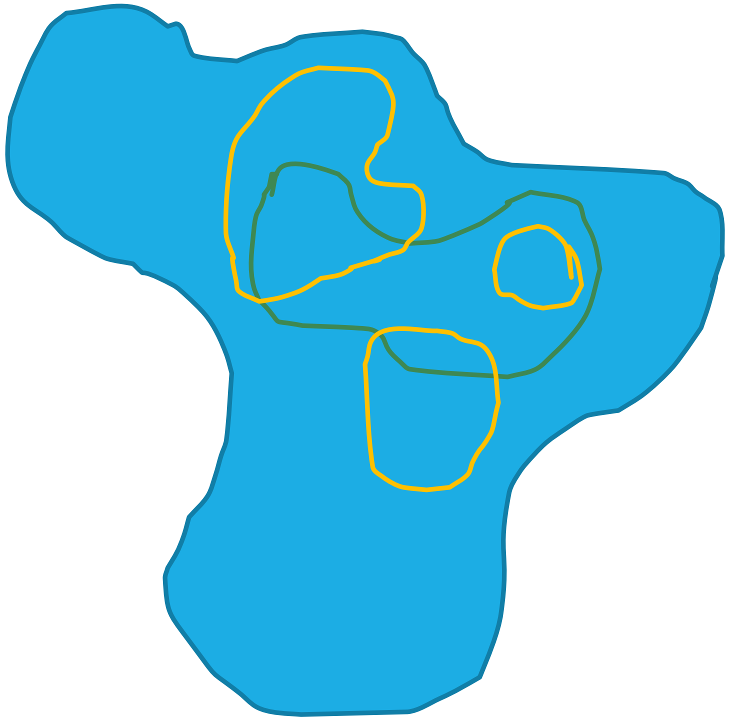
DEPENDENCY GRAPH

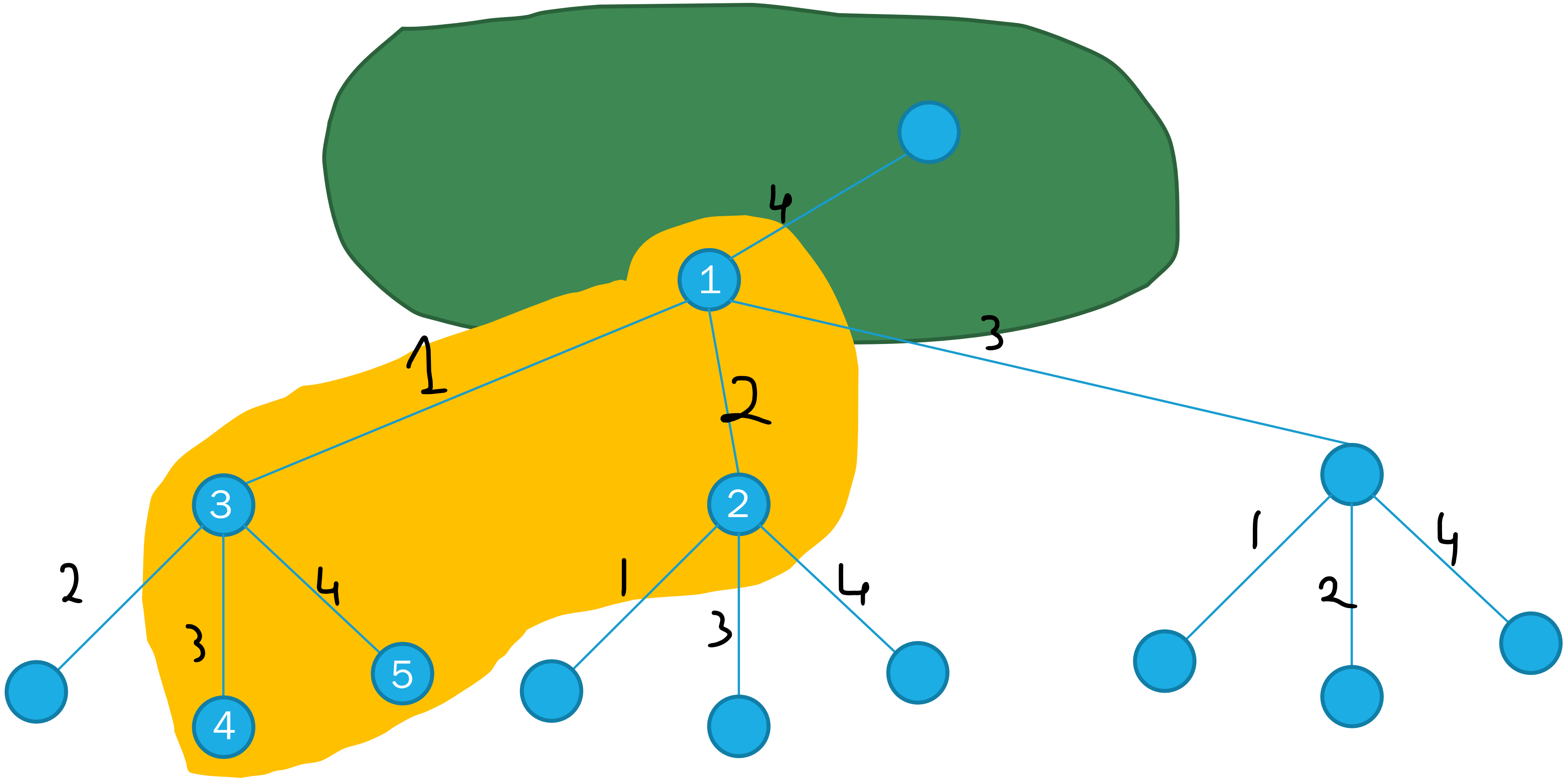
- Each vertex corresponds a bad event $B_{u,v}$, which happens with probability $\mathbb{P}\left[B_{u,v}\right] \leq d^{-6|S(u,v)|}$
  - Only when $S(u, v)$ form a connected component
- $B_{u,v}$ and $B_{u',v'}$ are independent if $S(u, v) \cap S(u', v') = \emptyset$
- How does the dependency graph look like?



DEPENDENCY GRAPH

- Let $B_{u,v}$ be such that $|S(u,v)| = k$
  - How many sets $S(u',v')$ of size l can depend on $B_{u,v}$?
    - At most $k\binom{d(l-1)}{l-1} \leq k(de)^{l-1}$





(1,2)

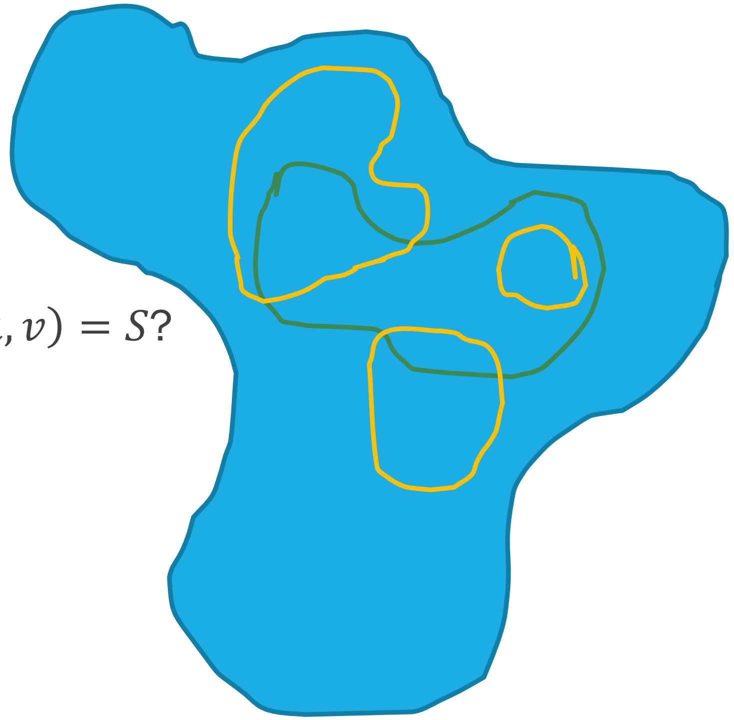
(1,1)

(3,3)

(3,4)

DEPENDENCY GRAPH

- Let $B_{u,v}$ be such that $|S(u,v)| = k$
  - How many sets $S(u',v')$ of size l can depend on $B_{u,v}$?
    - At most $k\binom{d(l-1)}{l-1} \leq k(de)^{l-1}$
- For a given support $S \subseteq [n]$, how many pairs of vectors u, v are there such that $S(u,v) = S$?
  - At most $3^{|S|}$
- Together, at most $3k(3de)^{l-1}$ neighbors of weight l



LOVASZ LOCAL LEMMA

- Let H be a dependency graph
  - $A_1 \dots A_n$ are vertices corresponding bad events
  - $A_i \sim A_j$ if A_i and A_j are dependent
  - Bounded degree d
- If $\mathbb{P}[A_i] \leq \frac{1}{4d}$ then $\mathbb{P}\left[\overline{A_1} \wedge \cdots \wedge \overline{A_n}\right] > 0$

- If we assign each event A_i a quantity $x_i \in [0,1]$ such that for all i we get $\mathbb{P}[A_i] \leq x_i \prod_{A_j \sim A_i}\left(1 - x_j\right)$

$$\mathbb{P}\left[\overline{A_1} \wedge \cdots \wedge \overline{A_n}\right] > \prod_{i=1}^{n}(1 - x_i) > 0$$

LOVASZ LOCAL LEMMA

- Choose $x_{u,v} = d^{-3k} = d^{-3|S(u,v)|}$

$$x_{u,v} \prod_{B_{u',v'} \sim B_{u,v}} (1 - x_{u',v'}) = d^{-3k} \prod_{B_{u',v'} \sim B_{u,v}} \left(1 - d^{-3|S(u',v')|}\right)$$

$$= d^{-3k} \prod_{l \in [n]} \left(1 - d^{-3l}\right)^{3k \cdot (3de)^{l-1}}$$

$$\geq d^{-3k} \exp\left(2 \cdot 3k \sum_{l \in [n]} d^{-3l} \cdot (3de)^{l-1}\right)$$

$$\geq d^{-3k} e^{-3k}$$

$$\geq d^{-6k}$$

- Thus, $\mathbb{P}\left[\wedge \overline{B_{u,v}}\right] > 0$

GOOD SIGNINGS EXIST

- $0 < \mathbb{P}\left[\wedge \overline{B_{u,v}}\right] = \mathbb{P}\left[\forall u, v \in \{0,1\}^n: \frac{u^t M_s v}{\|u\|\|v\|} \leq 10\sqrt{d \log(d)}\right]$
- There is a strictly positive probability for such a signing s
- Particularly, there exists such a signing, which finishes the proof

PROOF STEPS

- There exists a signing s such that $\frac{|u^t M_s v|}{\|u\|\|v\|} \leq 10\sqrt{d \log d}$ for all $u, v \in \{0,1\}^n$

- Let $A \in S^n(\mathbb{R})$ such that $\|A_i\|_1 \leq d$ and $|A_{i,i}| = O\left(\alpha\left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$.

  If for any two vectors $u, v \in \{0,1\}^n$ we have $\frac{u^t A v}{\|u\|\|v\|} \leq \alpha$

  Then the spectral radius of A is $O\left(\alpha\left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$

- Plug $\alpha = 10\sqrt{d \cdot \log(d)}$ to obtain the desired result

CONVERSE MIXING LEMMA

- The Expander Mixing Lemma states that for all $S, T \subseteq V$ holds

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|}$$

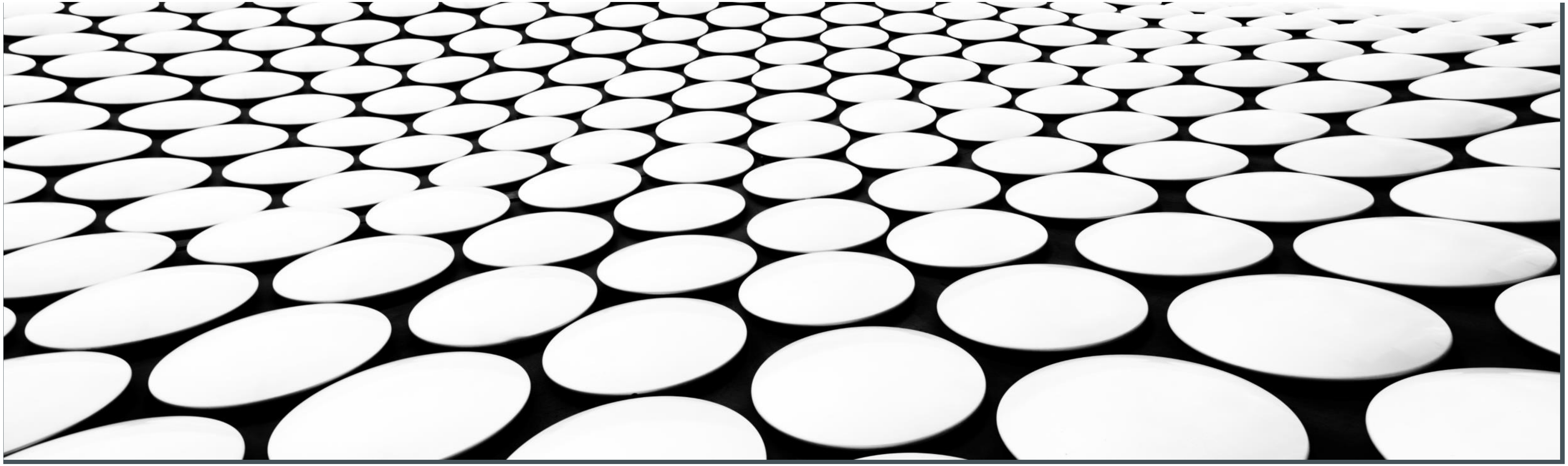
- Suppose we have for some graph G and some value $\alpha > 0$ that all $S, T \subseteq V$ with $S \cap T = \phi$ uphold

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \alpha\sqrt{|S||T|}$$

- We would like to say something about the spectral gap of G.

$$\frac{\left| u^t (M_G - \frac{d}{n}\mathbb{J}) v \right|}{\|u\|\|v\|} = \frac{\left| u^t M_G v - \frac{d}{n} u^t \mathbb{J} v \right|}{\|u\|\|v\|} = \frac{\left| e(S,T) - \frac{d}{n}|S||T| \right|}{\sqrt{|S||T|}} \leq \alpha$$

BILU – LINIAL EXPANDERS – CONSTRUCTION



WHAT NOW?

- A remark about a converse to the Expander Mixing Lemma
- Almost all 2-lifts are "good" (with assumptions on the initial graph)
  - So iteratively lifting yields a "good" expander of arbitrary size w.h.p.
- Derandomazation which gives a (weakly) explicit construction (deterministic, in polynomial time)

- If time permits:
  - A few words about trying to get a strongly explicit construction
  - A few words about the proof of the super technical lemma, and possible interesting byproducts

CONVERSE MIXING LEMMA

- The Expander Mixing Lemma states that for all $S, T \subseteq V$ holds

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|}$$

- Suppose we have for some graph G and some value $\alpha > 0$ that all $S, T \subseteq V$ with $S \cap T = \phi$ uphold

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \alpha\sqrt{|S||T|}$$

- We would like to say something about the spectral gap of G. Denote $\mathbb{1}_X$ to be the indicator function, O the all-ones matrix:

$$\alpha \geq \frac{\left| e(S,T) - \frac{d}{n}|S||T| \right|}{\sqrt{|S||T|}} = \frac{\left| \mathbb{1}_S^t M_G \mathbb{1}_T - \frac{d}{n}\mathbb{1}_S^t O \mathbb{1}_T \right|}{|\mathbb{1}_S||\mathbb{1}_T|} = \frac{\left| \mathbb{1}_S^t (M_G - \frac{d}{n} O)\mathbb{1}_T \right|}{|\mathbb{1}_S||\mathbb{1}_T|}$$

CONVERSE MIXING LEMMA

- Now we have $\alpha \geq \frac{\left|u^t\left(M_G - \frac{d}{n}O\right)v\right|}{|u||v|}$ for all $u, v \in \{0,1\}^n$ $with\ S(u) \cap S(v) = \phi$
- The lemma – "Let $A \in M_{n\times n}$ a real symmetric matrix such that the l_1 norm of each row is at most d, and all diagonal entries are, in absolute value, $O\left(\alpha * \left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$. Assume that for any $u, v \in \{0,1\}^n$ $with\ S(u) \cap S(v) = \phi$ we have $\frac{|u^t A v|}{|u||v|} \leq \alpha$, then all eigenvalues of A are $O\left(\alpha * \left(\log\left(\frac{d}{\alpha}\right) + 1\right)\right)$ "
- The matrix $\left(M_G - \frac{d}{n}O\right)$ upholds the assumptions of the lemma, so we get a bound on its eigenvalues.
- It shares all eigenvalues (except $\lambda_1 = d$) with M_G (*why?*)

MOST 2-LIFTS ARE GOOD

- We saw every d-regular graph G had a 2-lift $\hat{G}$ with all new eigenvalues $O\left(\sqrt{(dlog^3 d)}\right)$
- For construction to be feasible we want to be able to find good 2-lifts
- Would be nice if most 2-lifts were good
  - But we will have to add some restrictions on G.

- The intuition – condition that makes sure that u, v with small supports work.
  - (recall during the previous proof we saw the chance of each 'bad event' was $\leq d^{-6*|S(u,v)|}$, so we only need worry about small supports
- Definition: a graph G is $(\beta, t) - sparse$ if for any $u, v \in \{0,1\}^n$ with $|S(u,v)| \leq t$ holds $|u^t M_G v| \leq \beta * |u||v|$

MOST 2-LIFTS ARE GOOD

Working memory

- Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$).
  and let $\hat{G}$ be the 2-lift from a random signing s. So w.h.p. –

  1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\|\|v\|} \leq \gamma(d) = 10\sqrt{d \log d}$

     this means the eigenvalues of $\hat{G}$ will be bounded as before

  2. $\hat{G}\ is\ (\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

     this means we can continue inductively

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\|\|v\|} \leq \gamma(d)$
2. $\hat{G}$ is $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

Working memory

- First we prove (1)
- Recall the 'bad events' - $\mathbb{P}\left[\frac{|u^t M_s v|}{\|u\|\|v\|} > \gamma(d)\right] \leq d^{-6|S(u,v)|}$
- For a given value of k there are at most $n\binom{d(k-1)}{k-1} \leq n(de)^{k-1}$ connected subgraphs of size k in G
- Each of those might have as many as 3^k options for u, v
- If $k = |S(u,v)| \leq \lceil \log_2 n \rceil$ we have (1) for free (G is $(\gamma(d), \lceil \log_2 n \rceil) - sparse$)
- $Pr((1) fails) \leq \sum_{k > \lceil \log_2 n \rceil} 3^k n(de)^{k-1} * d^{-6k} \leq \frac{1}{2d} n^{-5 \log_2 d + 4.03}$
  $= O(\frac{1}{poly(n^{\log_2 d})})$

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\| \|v\|} \leq \gamma(d)$

2. $\hat{G}$ *is* $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- Now we want (2) – meaning $|u^t M_{\hat{G}} v| \leq \gamma(d) \|u\| \|v\|$ for all $u, v \in \{0,1\}^{2n}$ such that $|S(u, v)| \leq 1 + \lceil \log_2 n \rceil$
- Because we care about $|u^t M_{\hat{G}} v| / \|u\| \|v\|$ we may assume each vertex in S(u) is connected (in $\hat{G}$) to some vertex in S(v)
  - otherwise drop it to get a harder bar to clear
- $M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix}$, $M_G = M_1 + M_{-1}$, $M_s = M_1 - M_{-1}$

Working memory

New:

$S(u, v)$ *is connected in* $\hat{G}$

$|S(u, v)| \leq 1 + \lceil \log_2 n \rceil$

$M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix}$,

$M_G = M_1 + M_{-1}$,

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\|\|v\|} \leq \gamma(d)$

2. $\hat{G}\ is\ (\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

Working memory

$S(u, v)\ is\ connected\ in\ \hat{G}$

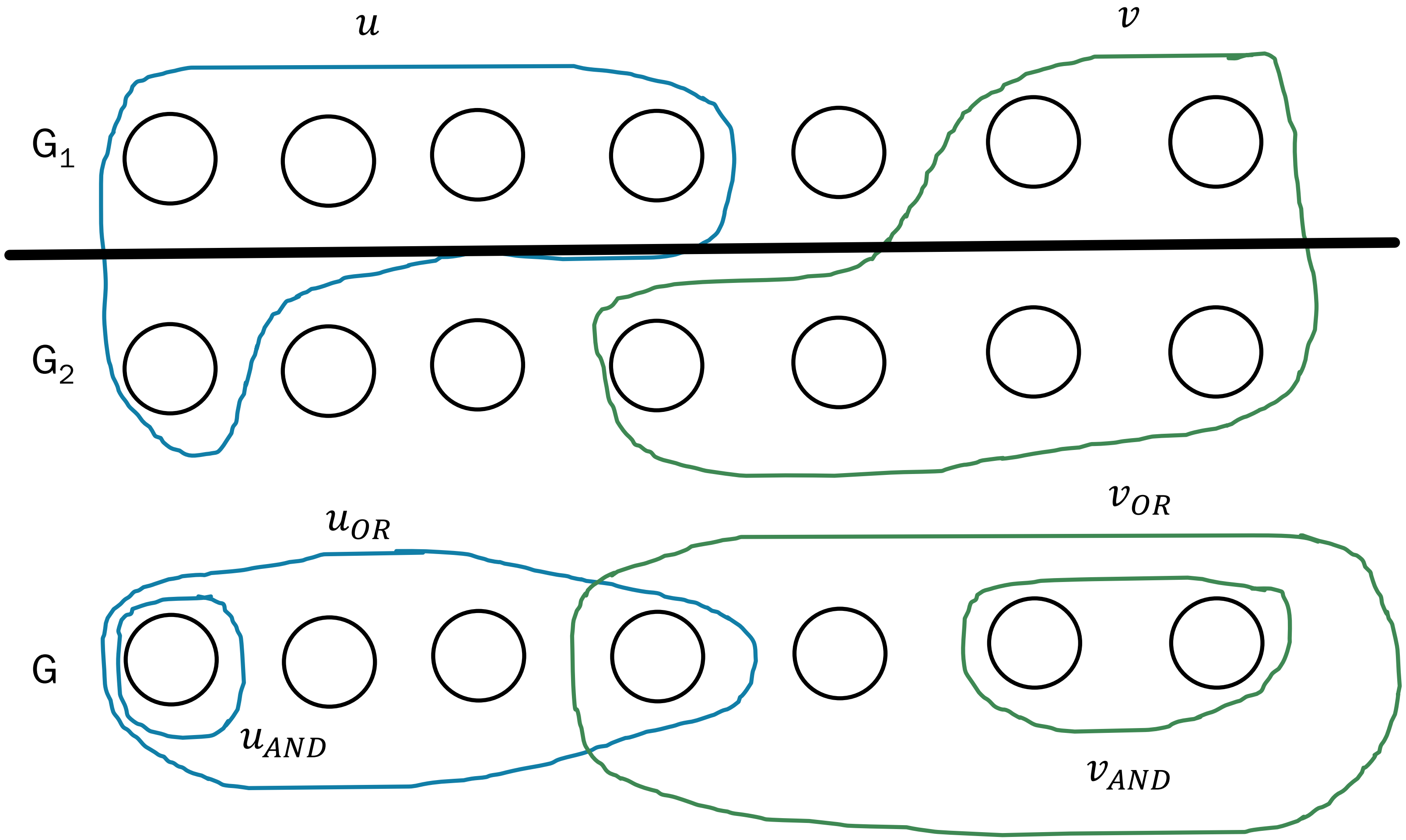
$|S(u, v)| \leq 1 + \lceil \log_2 n \rceil$

$M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix},$

$M_G = M_1 + M_{-1},$

- We would like to use our assumption that G is $(\gamma(d), \lceil \log_2 n \rceil) - sparse$
- For that we want subsets of V_G and not $V_{\hat{G}}$
  - Consider fibers in $\hat{G}$
- $u = (u_1, u_2) \quad u_{OR} = u_1 \vee u_2 \quad u_{AND} = u_1 \wedge u_2$

$$u = (u_1, u_2) \quad u_{OR} = u_1 \vee u_2 \quad u_{AND} = u_1 \wedge u_2$$



$$u = (u_1, u_2)$$

$$|u|^2 = |u_{OR}|^2 + |u_{AND}|^2$$

$$u_{OR} + u_{AND} = u_1 + u_2$$

$$|S(u, v)| \geq |S(u_{OR}, v_{OR})|$$

Note: $u, v \in \{0,1\}^{2n}$
$u_{OR}, v_{OR} \in \{0,1\}^{n}$

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\|\|v\|} \leq \gamma(d)$

2. $\hat{G}$ is $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- u_{OR}, u_{AND} will be our subsets in G for applying $(\gamma(d), \lceil \log_2 n \rceil) - sparseness$
- $|u|^2 = |u_{OR}|^2 + |u_{AND}|^2$; $u_{OR} + u_{AND} = u_1 + u_2$; $|S(u,v)| \geq |S(u_{OR}, v_{OR})|$

$u^t M_{\hat{G}} v = u_1^t M_1 v_1 + u_2^t M_{-1} v_1 + u_1^t M_{-1} v_2 + u_2^t M_1 v_2 \leq u_{OR}^t M_G v_{OR} + u_{AND}^t M_G v_{AND}$

- If $|S(u_{OR}, v_{OR})| \leq \lceil \log_2 n \rceil$ we get from G being $(\gamma(d), \lceil \log_2 n \rceil) - sparse$:
- $u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR} + u_{AND}^t M_G v_{AND} \leq \gamma(d) * (|u_{OR}||v_{OR}| + |u_{AND}||v_{AND}|)$ $\leq \gamma(d) * (|u||v|)$
- Which is precisely (2). Which leaves the case of $|S(u_{OR}, v_{OR})| \geq 1 + \lceil \log_2 n \rceil$

Working memory

$S(u,v)$ is $connected$ in $\hat{G}$

$|S(u,v)| \leq 1 + \lceil \log_2 n \rceil$

$M_{\hat{G}} = \begin{bmatrix} M_1 & M_{-1} \\ M_{-1} & M_1 \end{bmatrix}$

$M_G = M_1 + M_{-1},$

New:

$u = (u_1, u_2)$

$u_{OR} = u_1 \vee u_2$; $u_{AND} = u_1 \wedge u_2$

$|S(u,v)| \geq |S(u_{OR}, v_{OR})|$

$u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR} + u_{AND}^t M_G v_{AND}$

$|S(u_{OR}, v_{OR})| \geq 1 + \lceil \log_2 n \rceil$

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_S v|}{\|u\|\|v\|} \leq \gamma(d)$

2. $\hat{G}$ is $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- From $|S(u,v)| \geq |S(u_{OR}, v_{OR})|$, we get $|S(u,v)| = |S(u_{OR}, v_{OR})| = 1 + \lceil \log_2 n \rceil$
- So $S(u,v)$ cannot contain both vertices from a fiber of $\hat{G}$
  - In particular, $u_{AND} = v_{AND} = \vec{0}$ and so $|u| = |u_{OR}|$; $|v| = |v_{OR}|$
- We want to apply G's $(\gamma(d), \lceil \log_2 n \rceil)$-sparseness to $u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR}$
- Only problem is $|S(u_{OR}, v_{OR})|$ is too big by 1
- By taking one vertex out of $S(v)$ we get:
- $u_{OR}^t M_G v'_{OR} \leq \gamma(d) \sqrt{|S(u)||S(v) - 1|}$

Working memory

$S(u,v)$ is $connected$ in $\hat{G}$

$|S(u,v)| \leq 1 + \lceil \log_2 n \rceil$

$u = (u_1, u_2)$

$u_{OR} = u_1 \vee u_2$; $u_{AND} = u_1 \wedge u_2$

$|S(u,v)| \geq |S(u_{OR}, v_{OR})|$

$u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR} + u_{AND}^t M_G v_{AND}$

$|S(u_{OR}, v_{OR})| \geq 1 + \lceil \log_2 n \rceil$

New:

$|S(u,v)| = |S(u_{OR}, v_{OR})| = 1 + \lceil \log_2 n \rceil$

$|u| = |u_{OR}|$; $|v| = |v_{OR}|$

$u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR}$

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_s v|}{\|u\| \|v\|} \leq \gamma(d)$

2. $\hat{G}$ *is* $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

MOST 2-LIFTS ARE GOOD

- taking one vertex out of $S(v)$ gives – $u_{OR}^t M_G v'_{OR} \leq \gamma(d)\sqrt{|S(u)||S(v) - 1|}$
- Summing over all candidates for vertices to take out gives
  $$|\mathrm{S(v)} - 2| * u_{OR}^t M_G v_{OR} \leq |S(v)|\gamma(d)\sqrt{|S(u)||S(v) - 1|}$$
- We are trying to prove $u_{OR}^t M_G v_{OR} \leq \gamma(d)\frac{|S(v)|-1}{|S(v)|-2}\sqrt{|S(u_{OR})||S(v_{OR})|}$.
- So assume $u_{OR}^t M_G v_{OR} > \gamma(d)\sqrt{|S(u_{OR})||S(v_{OR})|}$, suppose $|S(u_{OR})| \leq |S(v_{OR})|$
- Regularity gives $u_{OR}^t M_G v_{OR} \leq d|S(u_{OR})|$
- Together we get $\frac{|S(u_{OR})|}{|\mathrm{S(v_{OR})}|} > \frac{\gamma(d)^2}{d^2} = \frac{100 \log_2 d}{d}$

Working memory

$S(u, v)$ *is connected in* $\hat{G}$

$u = (u_1, u_2) \qquad u_{OR} = u_1 \vee u_2$

$|S(u, v)| = |S(u_{OR}, v_{OR})| = 1 + \lceil \log_2 n \rceil$

$|u| = |u_{OR}|$; $|v| = |v_{OR}|$

$u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR}$

New:

$$1 \geq \frac{|S(u_{OR})|}{|S(v_{OR})|} > \frac{\gamma(d)^2}{d^2} = \frac{100 \log_2 d}{d}$$

$$u_{OR}^t M_G v_{OR} \leq \sqrt{\frac{|S(v)|}{|S(v)| - 1}} \frac{\gamma(d)(|S(v)| - 1)}{|S(v)| - 2} \sqrt{|S(u)||S(v)|}$$

$$|S(u_{OR})| \leq |S(v_{OR})| \geq \frac{\log_2 + 1}{2}$$

MOST 2-LIFTS ARE GOOD

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_S v|}{\|u\| \|v\|} \leq \gamma(d)$

2. $\hat{G}$ *is* $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- Edges in $\hat{G}$ from $S(u)$ to $S(v)$ come from edges in G between $S(u_{OR})$ and $S(v_{OR})$. Since no fiber is contained in $S(u, v)$ each edge survives with p=0.5
- $\mathbb{E}[u^t M_{\hat{G}} v] \leq \frac{1}{2} \mathbb{E}[u_{OR}^t M_G v_{OR}] \leq \left(\frac{1}{2}\sqrt{\frac{|S(v)|}{|S(v)|-1}}\right)\gamma(d)\sqrt{|S(u_{OR})||S(v_{OR})|}$
- By multiplicative Chernoff bound $\mathbb{P}(X > (1+\delta)\mu) \leq \exp\left(-\frac{\delta^2 \mu}{3}\right)$ $(\delta < 1)$
- $\mathbb{P}\left(u^t M_{\hat{G}} v > \gamma(d)\sqrt{|S(u)||S(v)|}\right) \leq \exp\left(-0.1\gamma(d)\sqrt{|S(u)||S(v)|}\right)$

  $\leq \exp(-0.05\gamma(d)(\log_2 n + 1)\left(10\sqrt{\frac{\log_2 d}{d}}\right) = \exp(-5 \log_2 d\,(1 + \log_2 n))$

Working memory

$S(u, v)$ *is connected in* $\hat{G}$

$u = (u_1, u_2) \qquad u_{OR} = u_1 \vee u_2$

$|S(u, v)| = |S(u_{OR}, v_{OR})| = 1 + \lceil \log_2 n \rceil$

$|u| = |u_{OR}|$; $|v| = |v_{OR}|$

$u^t M_{\hat{G}} v \leq u_{OR}^t M_G v_{OR}$

$\leq \sqrt{\frac{|S(v)|}{|S(v)| - 1}}\gamma(d)\sqrt{|S(u)||S(v)|}$

$\frac{|S(u_{OR})|}{|S(v_{OR})|} > \frac{100 \log_2 d}{d}$

$|S(u_{OR})| \leq |S(v_{OR})| \geq \frac{\log_2 + 1}{2}$

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ So w.h.p. –

(V) 1. for all $u, v \in \{0,1\}^n : \frac{|u^t M_S v|}{\|u\|\|v\|} \leq \gamma(d)$

2. $\hat{G}$ is $(\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

MOST 2-LIFTS ARE GOOD

Working memory

$S(u,v)$ is connected in $\hat{G}$

$|S(u,v)| = 1 + \lceil \log_2 n \rceil$

- For specific u,v with $|S(u,v)| = 1 + \lceil \log_2 n \rceil$ we got $\mathbb{P}\left(u^t M_{\hat{G}} v > \gamma(d)\sqrt{|S(u)||S(v)|}\right) < \exp(-5 \log_2 d \, (1 + \log_2 n))$
- There are at most $2n\binom{d(k-1)}{k-1} \leq 2n(de)^{k-1}$ connected subgraphs of size k in $\hat{G}$ and each has at most 3^k partitions to u,v. we care about $k = 1 + \lceil \log_2 n \rceil$
- Union bound gives us the chance of a signing to be bad is $O(\frac{1}{ploy(n^{\log_2 d})})$
- woohoo

FIRST ALGORITHM

- Start with $(\gamma(d), \lceil \log_2 n_0 \rceil) - sparse$ d-regular graph
- Pick a random signing and create the 2-lift graph
- Keep going until the graph is of the desired size
- Guaranteed to have $O\left(\frac{1}{poly(n^{\log_2 d})}\right)$ chance to end up with a bad graph
- Could repeat the process until you get a good one

DETERMINISTIC ALGORITHM

- Let G be d-regular and $(\gamma(d), \lceil\log_2 n\rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$). Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:

  1. the spectral radius of G_s is $O\left(\sqrt{(dlog^3 d)}\right)$

  2. $\hat{G}\ is\ (\gamma(d), 1 + \lceil\log_2 n\rceil) - sparse$

- We want to reconstruct our argument for bounding the probability of a 'bad' signing in a concise way so we can derandomize it (non of that "for all $u, v \in \{0,1\}^n$" stuff, too many of those...)
  - We swapped that out for a concise statement about the spectral radius of G_s

MOST 2-LIFTS ARE GOOD (AGAIN)

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$).
Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:

1. the spectral radius of G_s is $O\left(\sqrt{(dlog^3 d)}\right)$
2. $\hat{G}\ is\ (\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- For every <u>closed path</u> p <u>in G</u> of length $l = 2\lceil \log_2 n \rceil$ define $Y_p = \prod_{e \in p} s(e)$
- Name $\mathrm{Y} \coloneqq \sum_p Y_p$ and note that $\mathrm{Y} = Tr(M_s^l)$ to get
- $\mathbb{E}[Y] = \mathbb{E}[Tr(M_s^l)] = \mathbb{E}[\sum_{\mu \in spec(M_s)} \mu^l] = O\left(\sqrt{d(log_2 d)^3}\right)^l$
  - The last equation is due the bound we showed on the probability of bad μ.
  - we got $p \leq d^{-2l}$ and the worst an eigenvalue could be is d, so this is good enough to bound the effect on $\mathbb{E}[\mathrm{n} * \mu^l]$ to be <1
  - $\mathbb{E}[Y]$ is positive because l is even (why is it not identically zero?)
- Making sure Y is small will ensure (1)
  - By small we mean close to its expected value

Working memory
s is a random signing

New:
$l = 2\lceil \log_2 n \rceil$
$\mathbb{E}[\mathrm{Y}] = O\left(\sqrt{d(log_2 d)^3}\right)^l$
If Y is small, s upholds (1)

MOST 2-LIFTS ARE GOOD (AGAIN)

Let G be d-regular and $(\gamma(d), \lceil\log_2 n\rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$). Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:

1. the spectral radius of G_s is $O\left(\sqrt{(dlog^3 d)}\right)$
2. $\hat{G}$ is $(\gamma(d), 1 + \lceil\log_2 n\rceil) - sparse$

- Now we want to encode (2) as the expectancy of a random variable
- We saw that for $u, v \in \{0,1\}^{2n}$ $with$ $|S(u,v)| \le \lceil\log_2 n\rceil$ we get the sparseness condition directly from G being $(\gamma(d), \lceil\log_2 n\rceil) - sparse$. Also, $S(u,v)$ connected is good enough
- for u, v $with$ $S(u,v)$ $connected$ and $|S(u,v)| = 1 + \lceil\log_2 n\rceil$ define

$$Z_{u,v} = \begin{cases} d^l & u^t M_{\hat{G}} v > \gamma(d)\sqrt{|S(u)||S(v)|} \\ 0 & otherwise \end{cases}$$

- Want to bound $\mathbb{E}[Z]$ for $Z \coloneqq \sum Z_{u,v}$. We saw $P\left(Z_{u,v} \neq 0\right) < \exp\left(-5\log_2 d\,(1 + \log_2 n)\right)$
  - Using the whole u_{OR}, v_{OR} mess, all the assumptions still hold, this bound holds for any such u,v
- The number of (u,v) pairs is at most $2n * 3(3ed)^{\log_2 n}$ so by the union bound we get
- $\mathbb{E}[Z] < 2n * 3(3ed)^{\log_2 n} * \mathrm{d}^{\left(\left(2-\frac{5}{\ln(2)}\right)\log_2 n\right)} < d^{-3\log_2 n}$

Working memory
s is a random signing
$l = 2\lceil\log_2 n\rceil$
$\mathbb{E}[Y] = O\left(\sqrt{d(log_2 d)^3}\right)^l$
If Y is small, s upholds (1)
New:
$\mathbb{E}[Z] < d^{-3\log_2 n}$
If $Z = 0$, s upholds (2)
If $Z \neq 0$, $Z \ge d^{2\lceil\log_2 n\rceil}$

MOST 2-LIFTS ARE GOOD (AGAIN)

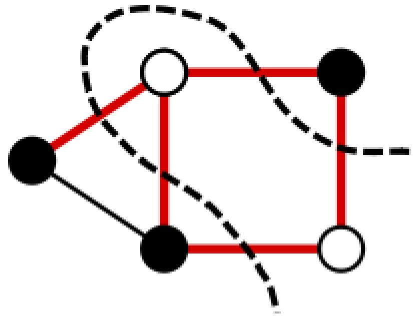
Let G be d-regular and $(\gamma(d), \lceil\log_2 n\rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$).
Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:

1. the spectral radius of G_s is $O\left(\sqrt{(dlog^3 d)}\right)$
2. $\hat{G}$ is $(\gamma(d), 1 + \lceil\log_2 n\rceil) - sparse$

- Now combine the whole thing into one random variable
- Let $X = Y + Z$ clearly $\mathbb{E}[Y] \approx \mathbb{E}[X] < d^{2\lceil\log_2 n\rceil}$
- So if X is less than it's expected value, $Z = 0$ and $Y = X \leq \mathbb{E}[X] \approx \mathbb{E}[Y]$
- So if $X \leq \mathbb{E}[X]$ we know that (1) and (2) hold
- All that remains is finding a signing which gives a small value of X
- Note that Y and Z are each a sum of $poly(n^{\log_2 d}) - many$ random variables, and the conditional expectation of each one (if we pre-assign some of the edges of s) can be computed in ploy-time

Working memory
s is a random signing
$l = 2\lceil\log_2 n\rceil$
$\mathbb{E}[Y] = O\left(\sqrt{d(log_2 d)^3}\right)^{2\lceil\log_2 n\rceil}$
If Y is small, s upholds (1)
$\mathbb{E}[Z] < d^{-3\log_2 n}$
If $Z = 0$, s upholds (2)
If $Z \neq 0$, $Z \geq d^{2\lceil\log_2 n\rceil}$
New:
$\mathbb{E}[Y_p], \mathbb{E}[Z_{u,v}]$ can be computed in ploy-time even under conditional partial s
if $X \leq \mathbb{E}[X]$ (1) and (2) hold

SHORT DETOUR - MAX CUT



- Given graph G, find $S \subseteq V$ such that $e(S, V \setminus S)$ is maximal (weighted case - $\sum_{e \in e(S, V \setminus S)} w_e$
- NP-hard
- Can be randomly approximated:
  - Put each vertex in S or V\S with probability 0.5
  - $E[size\ of\ cut] = \frac{1}{2} \sum_e w_e$
- This algorithm can be derandomized!

DERANDOMAZATION – THE METHOD OF CONDITIONAL EXPECTATION

- At each step in the algorithm, what will the expected size of the final cut be?
- $\mathbb{E}[final\ cut] = \sum_{e\ already\ in\ cut} w_e + \frac{1}{2}\sum_{e\ is\ undecided} w_e$
- This can be computed in poly time
- Note that for each unassigned vertex v we have:
- $\mathbb{E}[final\ cut] = \frac{1}{2}\mathbb{E}[final\ cut|v\ in\ S] + \frac{1}{2}\mathbb{E}[final\ cut|v\ in\ V\backslash S]$
- So at least one of these is no less than $\mathbb{E}[final\ cut]$.
- Place v so that the expected size of the final cut is largest, and continue to the next vertex
- We are guaranteed to get a cut of size $\geq \mathbb{E}[final\ cut|nothing\ assigned\ yet] = \frac{1}{2}\sum w_e$

DERANDOMAZATION

Let G be d-regular and $(\gamma(d), \lceil \log_2 n \rceil) - sparse$ (with $\gamma(d) = 10\sqrt{dlog_2 d}$).
Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:

1. the spectral radius of G_s is $O\left(\sqrt{(dlog^3 d)}\right)$
2. $\hat{G}\ is\ (\gamma(d), 1 + \lceil \log_2 n \rceil) - sparse$

- We had $X = Y + Z$ such that if $X \leq \mathbb{E}[X]$ (1) and (2) hold
- $Y = \sum_p Y_p\ ; Z = \sum_{u,v} Z_{u,v}$
- $\mathbb{E}[Y_p], \mathbb{E}[Z_{u,v}]$ can be computed in ploy-time even under conditional partial s
- For each edge e:
  - Compute expected value of X if $s(e) = 1$ and if $s(e) = -1$
  - Give e the sign which yields a lower expected value of X
  - Continue to next edge
- The signing you get will be good (deterministically!)

Working memory
s is a random signing
$\mathbb{E}[Y_p], \mathbb{E}[Z_{u,v}]$ can be computed in ploy-time even under conditional partial s
if $X \leq \mathbb{E}[X]$ (1) and (2) hold

ON AN UNRELATED NOTE (1)

- The construction is weakly explicit but not strongly explicit.
- Turns out there's a construction be Naor and Naor which lets us create a $(k, \epsilon) - wise\ independent$ sample space.
- With parameter choice of $k = dlog_2 n$, $\epsilon = d^{-2dlog_2 n}$ the size of the sample space will be $poly(n^{\log_2 d})$
- Using these random coin-tosses for giving signs to the edges gives chance of failure will be $\leq 1/n$
  - Won't go into more details, though there are plenty

ON AN UNRELATED NOTE (2)

- The super-technical lemma states - "Let $A \in M_{n\times n}$ a real symmetric matrix such that the l_1 norm of each row is at most d, and all diagonal entries are, in absolute value, $O\left(\alpha * \left(\log\left(\frac{d}{\alpha}\right)+1\right)\right)$. Assume that for any $u, v \in \{0,1\}^n$ $with\ S(u) \cap S(v) = \phi$ we have $\frac{|u^t A v|}{|u||v|} \le \alpha$, then all eigenvalues of A are $O\left(\alpha * \left(\log\left(\frac{d}{\alpha}\right)+1\right)\right)$ "
- It proves this claim by taking any vector x, and bounding its Rayleigh Quotient by splitting it into a sum of several multiples of 0,1-vectors (by the range of the entry size – 0.5-1, 0.25-0.5, 0.125-0.25 and so on) and using the lemma's assumption on each pair of those.
- This lets us get from a large eigenvalue to a counter-example of combinatorial edge expansion (S and T such that $\left|e(S,T) - \frac{d}{n}|S||T|\right|$ is large. (note that we lose a factor of log)
  - This is cool (and could maybe even be useful for someone at some point? But mostly cool)

QUESTIONS?