Valuations

Gil Cohen

May 27, 2019

Definition

Let L be a field. A (discrete) valuation is a map

$$v \colon L^{\times} \to \mathbb{Z}$$

s.t. $\forall x, y \in L^{\times}$:

- $v(xy) = v(x) + v(y)$ (namely, v is a group homomorphism).
- $v(x + y) \geq \min(v(x), v(y))$.

We extend v to L by defining $v(0) = +\infty$ ("larger than everything").

Remark

- *$v \equiv 0$ is called the trivial valuation.*
- *If v is a valuation and $n \in \mathbb{N}$ then nv is a valuation.*
- *v surjective $\iff \exists x \in L^{\times}$ such that $v(x) = 1$.*

The notion of valuation is motivated by "expressing multiplicities" in factorization.

Example

Let A be a Dedekind domain with $K = \mathrm{Frac}(A)$. Let $P \in \mathrm{Max}(A)$. We associate with P the P-adic valuation

$$v_P \colon K^\times \to \mathbb{Z}$$

as follows: For $0 \neq x \in A$, we factor the ideal $\langle x \rangle$ in A:

$$\langle x \rangle = \prod_{P \in \mathrm{Max}(A)} P^{\mathrm{ord}_P(x)},$$

and define $v_P(x) = \mathrm{ord}_P(x)$. We extend v_P to K in the unique possible way, namely, if $0 \neq x \in K$ then $x = \frac{a}{b}$ for $a, b \in A$ and so we define $v_P(x) = v_P(a) - v_P(b)$. Check this is well-defined.

Remark

- v_P *is a surjective valuation as* P^2 *is strictly contained in* P*.*
- $v_P(x) \geq 0$ *for all* $x \in A$*.*
- $v_P(x) = 0$ *for* $x \in A \iff x \notin P$*.*
- $P \neq Q \in \mathrm{Max}(A) \implies v_P \neq v_Q$*. Indeed, take* $x \in P \setminus Q$*, then* $v_P(x) > 0$ *whereas* $v_Q(x) = 0$*.*

Claim

Let L be a field and $v : L^\times \to \mathbb{Z}$ a valuation. Then,

$$v(1) = v(-1) = 0$$
$$v(-x) = v(x)$$
$$v\left(\frac{1}{x}\right) = -v(x)$$

Proof.

$$v(1) = v(1 \cdot 1) = v(1) + v(1) \quad \Longrightarrow \quad v(1) = 0.$$

$$0 = v(1) = v((-1)\cdot(-1)) = 2v(-1) \quad \Longrightarrow \quad v(-1) = 0.$$

Take $x \in L^\times$. We have that

$$v(-x) = v((-1)x) = v(-1) + v(x) = v(x).$$

□

Proof.

As for the last item,

$$0 = v(1) = v\left(x \cdot \frac{1}{x}\right) = v(x) + v\left(\frac{1}{x}\right)$$

□

Claim (Strict triangle inequality)

Let L be a field and $v : L^\times \to \mathbb{Z}$ a valuation. Let $x, y \in L^\times$ with $v(x) \neq v(y)$. Then,

$$v(x+y) = \min(v(x), v(y)).$$

Proof.

Assume wlog $v(x) < v(y)$. Assume towards a contradiction that $v(x+y) > v(x)$. Then,

$$v(x) = v(x+y-y) \geq \min(v(x+y), v(-y)) > v(x)$$

contradiction. □

Example

Let K be a field, $A = K[x]$ and $L = K(x)$. The maximal ideals in $K[x]$ are in bijection with monic irreducible polynomials over K. Thus, we have a distinct valuation $v_{p(x)} : K(x)^\times \to \mathbb{Z}$ for every monic irreducible polynomial in $K[x]$.

We can point at one more valuation

$$v_\infty \left(\frac{f(x)}{g(x)} \right) = \deg(g) - \deg(f).$$

Claim

The valuation v_∞ of $K(x)$ is equal to the valuation v_P of $K(x)$ associated to the maximal ideal $P = \frac{1}{x}K[\frac{1}{x}]$ of the subring $K[\frac{1}{x}]$ of $K(x)$.

Definition

Let $v : K^\times \to \mathbb{Z}$ be a valuation. Define

$$\mathcal{O}_v = \left\{x \in K^\times \mid v(x) \geq 0\right\} \cup \{0\},$$
$$\mathcal{M}_v = \left\{x \in K^\times \mid v(x) > 0\right\} \cup \{0\}.$$

The ring (to be proven) $\mathcal{O}_v$, associated with the valuation v is called a discrete valuation ring (DVR).

Claim

$\mathcal{O}_v$ is a local subring of K with maximal ideal $\mathcal{M}_v$.

Proof

$\mathcal{O}_v$ **is a ring.** *Take* $x, y \in \mathcal{O}_v$ *then* $v(x), v(y) \geq 0$ *and so*

$$v(xy) = v(x) + v(y) \geq 0$$
$$v(x + y) \geq \min(v(x), v(y)) \geq 0.$$

Furthermore, $v(1) = 0$ *and so* $1 \in \mathcal{O}_v$*.*

$\mathcal{M}_v$ **is an ideal of** $\mathcal{O}_v$**.** *Indeed,* $\forall m, m' \in \mathcal{M}_v, x \in \mathcal{O}_v$

$$v(m + m') \geq \min(v(m), v(m')) > 0$$
$$v(mx) = v(m) + v(x) > 0$$

Proof.

$\mathcal{O}_v$ **is local.** Recall that $\forall x \in K^\times$ we have that

$$v\left(\frac{1}{x}\right) = -v(x).$$

Thus,

$$x \in \mathcal{O}_v^\times \iff v(x) = 0 \iff x \notin \mathcal{M}_v$$

and so $\mathcal{O}_v$ is local. □

Another property of valuations is the following.

Claim

Let $v : L^\times \to \mathbb{Z}$ be a nontrivial valuation. Let $\pi \in \mathcal{M}_v$ be an element with minimal value $c = v(\pi)$. Then $c \mid v(x)$ for all $x \in L^\times$.

Proof.

Take $x \in \mathcal{O}_v$. if c does not divide $v(x)$ then $v(x) = cq + r$ with $0 < r < c$. Thus,

$$v\left(\frac{x}{\pi^q}\right) = v(x) - qv(\pi) = r,$$

contradicting the minimality of c. For $x \notin \mathcal{O}_v$ we have $v(x) < 0$ and so $-v(x) = v(\frac{1}{x}) > 0$. Hence, $c \mid v(x)$. □

Corollary

Let L be a field and $v : L^\times \to \mathbb{Z}$ be a nontrivial valuation. Then, $\exists c \in \mathbb{N}$ such that $v/c : L^\times \to \mathbb{Z}$ is a surjective valuation.

Claim

Let L be a field and $v : L^\times \to \mathbb{Z}$ be a valuation. Let $\pi \in \mathcal{M}_v$ be an element with minimal value $v(\pi)$. Then, every element $x \in L^\times$ can be written as $x = u\pi^n$ with $u \in \mathcal{O}_v^\times$. Moreover, n is unique.

Proof.

Take $x \in L^\times$. By the previous claim, $\exists n \in \mathbb{N} \quad v(x) = n \cdot v(\pi)$ and so

$$v(x) = v(\pi^n) \implies v(x/\pi^n) = 0 \implies x/\pi^n \in \mathcal{O}_v^\times.$$

Hence, $\exists u \in \mathcal{O}_v^\times$ s.t. $x = u\pi^n$.

As for uniqueness, if $v(u\pi^c) = v(w\pi^d)$ then

$$0 = v(u) - v(w) = v(\pi^{d-c}) = (d-c)v(\pi) \implies d = c$$

□

Claim

Let K be a field and $v\colon K^\times \to \mathbb{Z}$ a nontrivial valuation. Then, $\mathcal{O}_v$ is a local PID.

Proof.

Given an ideal I of $\mathcal{O}_v$ let $d \geq 1$ be the minimal integer such that $\pi^d \in I$. We claim that

$$I = \pi^d \mathcal{O}_v.$$

Clearly $\pi^d \mathcal{O}_v \subseteq I$. Now, if $x \in I$ then $x = u\pi^c$ for $c \geq d$ and $u \in \mathcal{O}_v^\times$. Thus, $x = (u\pi^{c-d})\pi^d$. Since $u\pi^{c-d} \in \mathcal{O}_v$ we conclude $x \in \pi^d \mathcal{O}_v$. □

Corollary

Let $v : L^\times \to \mathbb{Z}$ *be a nontrivial valuation. Let* $\pi \in \mathcal{M}_v$ *be an element with minimal value. Then,*

- $\mathcal{M}_v = \pi \mathcal{O}_v$.
- v *is uniquely determined by* $v(\pi)$.
- v *is surjective* $\iff$ $v(\pi) = 1$.

Definition

For a field K we let

$$\text{SurjVal} = \left\{ v : K^\times \twoheadrightarrow \mathbb{Z} \right\}$$

Definition

Let K be a field. We define

$$\text{LPID} = \{ A \subseteq K \text{ local PID with } \text{Frac}(A) = K \}$$

Claim

Let K be a field. The map

$$\begin{aligned} \text{SurjVal} &\to \text{LPID} \\ v &\mapsto \mathcal{O}_v \end{aligned}$$

is a bijection.

Proof

We proved that this map is well-defined.

Injectivity. *Take* $v_1, v_2 \in \mathrm{SurjVal}$ *with* $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$. *Then,* $\mathcal{M}_{v_1} = \mathcal{M}_{v_2}$ *and so if* π *is a generator for this maximal ideal, then* $v_1(\pi) = v_2(\pi) = 1$ *by surjectivity. The proof then follows since* v_1, v_2 *are determined by their value on* π.

Subjectivity. *Given* $\mathcal{O} \in \mathrm{LPID}$ *let* $\mathcal{M}$ *be its unique maximal ideal.* $\mathcal{O}$ *is a Dedekind domain and so the* $\mathcal{M}$*-adic valuation* $v_{\mathcal{M}}$ *is well-defined. Since* $\mathrm{Frac}(\mathcal{O}) = K$ *the domain of* $v_{\mathcal{M}}$ *is* $K^{\times}$. *Check that* $v_{\mathcal{M}}$ *is a preimage of* $\mathcal{M}$ *under the map.*

Claim

Let A be a domain of dimension 1 with $\mathrm{Frac}(A) = K$. Then, the map

$$\begin{aligned} \left\{v : K^\times \twoheadrightarrow \mathbb{Z} \mid v(A) \geq 0\right\} &\to \mathrm{Max}(A) \\ v &\mapsto \mathcal{M}_v \cap A \end{aligned}$$

is well-defined. Furthermore, if A is a Dedekind domain then the map is a bijection.

Proof

Take $v : K^\times \twoheadrightarrow \mathbb{Z}$ *with* $v(A) \geq 0$*. Then,* $A \subseteq \mathcal{O}_v$*. Since* $M_v \in \mathrm{Max}(\mathcal{O}_v)$ *we have that*

$$M = \mathcal{M}_v \cap A \in \mathrm{Spec}(A).$$

Since $\dim(A) = 1$*, either* $M = \langle 0 \rangle$ *or* $M \in \mathrm{Max}(A)$*. We are ought to show that* $M \neq \langle 0 \rangle$*.*

Since $A \setminus M \subseteq \mathcal{O}_v^\times$ *we have that* $A_M \subseteq \mathcal{O}_v$*. However, if* $M = \langle 0 \rangle$ *then* $A_M = K \implies K = \mathcal{O}_v$ *implying* v *is trivial (and so not surjective).*

Proof.

We turn to prove that if A is a Dedekind domain then the map $v \mapsto \mathcal{M}_v \cap A$ is a bijection.
Fix $M \in \mathrm{Max}(A)$. Recall the M-adic valuation $v_M : K^\times \twoheadrightarrow \mathbb{Z}$ that we defined for Dedekind domains. We already noted that:

- v_M is surjective.
- $v_M(A) \geq 0$. Hence, $M \mapsto v_M$ is indeed a map $\mathrm{Max}(A) \to \{v : K^\times \twoheadrightarrow \mathbb{Z} \mid v(A) \geq 0\}$.
- $v_M \neq v_N$ for distinct $N, M \in \mathrm{Max}(A)$. Thus, $m \mapsto v_M$ is injective.
- Showing that the two maps above are inverses of each other is left as an exercise.

□

Corollary

Let A be a Dedekind domain with $\mathrm{Frac}(A) = K$. *Then,*

$$\bigcap_{v \mid v(A) \geq 0} \mathcal{O}_v = A.$$

Proof.

Recall that

$$A = \bigcap_{M \in \mathrm{Max}(A)} A_M.$$

The proof follows since there is a bijection between surjective valuations $v : K^\times \twoheadrightarrow \mathbb{Z}$ with $v(A) \geq 0$ and $\mathrm{Max}(A)$, where $\mathcal{O}_v = A_M$. Note that every valuation can be made surjective while keeping the property $v(A) \geq 0$.

□