

Exercise 4: Teleportation and Non-Local Games

1. (a) Draw the overall circuit for the teleportation protocol. In order to draw a classically-controlled gate (meaning a gate that is acting only if a measurement's outcome is a specific outcome, usually $|1\rangle$), use a meter and a double-wire (see [Figure 4.1](#)).

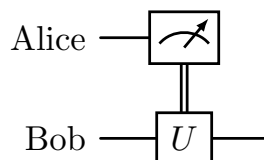


Figure 4.1: Circuit for teleportation of one qubit state from Alice to Bob, who share one EPR pair in advance.

- (b) Alice measures both her qubits after some unitary transformation, or said differently, she performs a complete measurement of her 4-dimensional quantum space in a different basis than the computational one. What is the basis in which she measures? If we think of them as two separate partial measurements, what are they? How do they correspond to the gates Bob has to apply, depending on their outcomes?
 - (c) Look at the projections of the initial state $|\psi\rangle_1 |EPR\rangle_{2,3}$ ¹ onto the basis you found in the previous subquestion and use this to reprove the correctness of the teleportation protocol.
2. (a) Alice knows she is holding a state of the form $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. Show how she can teleport $|\psi\rangle$ to Bob using 2 classical bits of communication, assuming they share an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

¹the subscripts help to track the different registers and prevent misunderstanding without the use of the symbol $\otimes$

- (b) Alice holds $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$. Show how she can teleport $|\psi\rangle$ to Bob using 4 classical bits of communication, assuming they share two EPR pairs. What is the generalization to n qubits?

4.1 CHSH Game

A strategy for a game is a set of probability distributions over the possible answers $A \times B$, for any given pair of questions $\forall x \in X, y \in Y, p(\cdot, \cdot | x, y) : A \times B \rightarrow [0, 1], \sum_{a \in A} \sum_{b \in B} p(a, b | x, y) = 1$. A strategy is called:

- Deterministic if given the questions the answers are uniquely determined $p(a, b | x, y) \in \{0, 1\}$. Equivalently, there are functions $f_A : X \times Y \rightarrow A, f_B : X \times Y \rightarrow B$ such that $p(a, b | x, y) = \delta_{f_A(x, y)=a} \cdot \delta_{f_B(x, y)=b}$.
- Classical if there is a random variable r on a probability space (Ω, μ) ², that should be thought of as shared randomness between Alice and Bob, such that Alice's answer is a function depending on her question and the random variable ($f_{A,x} := f_A(x, \cdot)$ measurable for any fixed question $x \in X$), and Bob's is a (similarly measurable) function depending on his question and the random variable: $f_A : X \times \Omega \rightarrow A, f_B : Y \times \Omega \rightarrow B$, $p(a, b | x, y) = \mu(\{\omega : f_A(x, r) = a, f_B(y, r) = b\}) = \mu(f_{A,x}^{-1}(a) \cap f_{B,y}^{-1}(b))$.
- Quantum if there is a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and for every question for Alice $x \in X$ there is a measurement on $\mathcal{H}_A$, $\{\Pi_a^x\}_{a \in A}$ and for every question for Bob $y \in Y$ a measurement on $\mathcal{H}_B$, $\{\Pi_b^y\}_{b \in B}$ such that the outcomes are their answers $p(a, b | x, y) = \langle \psi | \Pi_a^x \otimes \Pi_b^y | \psi \rangle$.
- Non-Signalling if Alice and Bob cannot use it to get any information about the other's question. To formalise it the strategy is required to have the same marginal distribution for an actor with any question for the other actor: $\forall x \in X, y, y' \in Y, a_0 \in A : \sum_{b \in B} p(a_0, b | x, y) = \sum_{b \in B} p(a_0, b | x, y')$ and similarly $\forall x, x' \in X, y \in Y, b_0 \in B : \sum_{a \in A} p(a, b_0 | x, y) = \sum_{a \in A} p(a, b_0 | x', y)$.

Clearly a classical strategy is a quantum one, and a quantum strategy is a non-signalling one. Note that a classical strategy that is deterministic has functions that are dependent only on the actor's question $f_A : X \rightarrow A, f_B : Y \rightarrow B$.

1. In this question we will explore the strength of non-signalling strategies:
 - (a) Prove that a non-signalling strategy that is deterministic is also classical.
 - (b) Is the following strategy for the CHSH game non-signalling?
If at least one of the players gets the question 0, they both answer with 0. If they both get 1, with probability 1/2 Alice answers 0 and Bob 1 and with probability 1/2 she answers 1 and he 0.

²We can just consider r as some n uniformly random bits

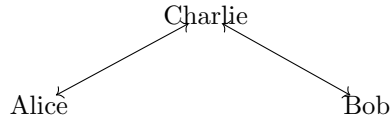


Figure 4.2: The connections between the three players in the teleported CHSH game. For quantum strategy the arrows represent shared quantum states. For 2-local non-signalling strategy they represent non-signalling devices for two sides/players.

- (c) Give a non-signalling strategy for the CHSH game with perfect winning probability 1. What does that imply regarding the containment of quantum strategies in non-signalling strategies?
2. In this question we will introduce another player, Charlie, and see a scenario in which a quantum strategy is stronger than a strategy that is non-local but restricted to be shared only between two players, in a specific configuration. For that we will also change a bit the way we think of the winning probability.

Alice and Bob are playing the CHSH game, and Charlie has only one possible question (equivalently, the referee does not send anything to Charlie) and needs to answer with {success, failure}. The winning probability will be defined as the probability of Alice and Bob to win the CHSH conditioned on Charlie answering “success”

$$\Pr[\text{Alice and Bob wins the CHSH} \mid \text{Charlie answers “success”}]$$

. In other words the case where Charlie answers “failure” does not get into the calculation of the winning probability. Charlie will be connected in a pair to both Alice and Bob. Each of these two pairs will either share a quantum state or have a device that produces correlations according to some non-signalling strategy (see [Figure 4.2](#)).

- (a) Calculate the winning probability (as defined above) for the following quantum strategy: Alice and Charlie share an EPR pair, and so do Bob and Charlie (the arrows in [Figure 4.2](#) are both EPR pairs). Charlie measures the two qubits, which are halves of the two EPR pairs, in a basis that includes the EPR pair. If the outcome is EPR then Charlie’s answer is “success” and otherwise it is “failure”. Alice and Bob use the measurements of the optimal quantum strategy we saw in the lecture (even though their qubits are initially parts of EPR pairs with Charlie, and are not entangled between them).
- (b) Let there be a non-signalling device (a non-local strategy of a two-player game with whatever question and answer sets) which has only one question for one of its sides (players), meaning this side has no choice of input/question. Prove that in this case the device can be

replaced by shared randomness, i.e. there is a random variable r such that both sides' answers are functions of their own question and the random variable.

- (c) Conclude the best winning probability (conditioned on Charlie's "success" answer) of the game, using a non-signalling device between Charlie and Alice and another device between Charlie and Bob (but no device is shared between Alice and Bob). Compare it to the winning probability of the quantum strategy with Charlie-Alice and Charlie-Bob shared states.
- (d) We would like to change the setting to have a more standard winning probability concept. For that we will change Charlie's possible answers and the predicate that dictates whether the game was won or lost. Charlie's answers will be two bits $c_1, c_2 \in \{0, 1\}$ and in order to win, when Alice got the question $x \in \{0, 1\}$ Bob $y \in \{0, 1\}$, and they answered $a, b \in \{0, 1\}$, the following equation must hold: $a \oplus b \oplus (xc_1) \oplus ((1-x)c_2) = xy$. In the quantum case we will still be using two EPR pairs, for Alice-Charlie and Bob-Charlie. Alice and Bob will still be using the measurements and answers according to the quantum strategy of CHSH. Charlie will be measuring in the basis you found in question item 1b in the Teleportation part of this exercise. Define what should be the corresponding answer for every basis-element outcome of Charlie's measurement so that the winning probability of this strategy will remain as in the first subquestion (a) (but this time without the conditional way we defined it there). Explain why this is called the teleported CHSH game.

hint: Charlie's answers are replacing the corrections Alice would have made, if Alice and Charlie were allowed to talk.