Explicit Almost Ramanujan Graphs via the Wide Replacement Product

Loosly following the original Ben-Aroya Ta-Shma paper

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Overview

1. The Zig-Zag product and entropy waves
2. What fails in doing three H steps?
3. The wide replacement product - three H steps.
4. The wide replacement product general parameters.

Entropy waves

Let us take a second look at the Zig-Zag product. Say (G, H) is a distribution over vertices. Let $\mathbf{x}$ be the "error vector" measuring the distance of (G, H) from uniform (note $\mathbf{x} \perp \mathbf{1}$).

Consider first the case in which $\forall g \in G \;\; H \mid (G = g) \approx U$. Algebraically, this is captured by $\mathbf{x} = \mathbf{x}_0 = \mathbf{y} \otimes \mathbf{u}$. Then,

1. The first $\tilde{\mathbf{W}}_H$ is "wasted" as $\mathbf{x}_1 = \tilde{\mathbf{W}}_H \mathbf{x}_0 = \mathbf{x}_0$.
2. The $\mathbf{P}$ step is a permutation, so the entropy of (G, H) does not change. However, G's entropy increases thus H's entropy decreases. Algebraically, observe that $(\mathbf{P}\mathbf{x}_1)^{\parallel} = (\mathbf{P}(\mathbf{y} \otimes \mathbf{u}))^{\parallel} = (\mathbf{W}_G \mathbf{y}) \otimes \mathbf{u}$. Hence,

$$\|(\mathbf{P}\mathbf{x}_1)^{\parallel}\| = \|\mathbf{W}_G \mathbf{y}\| \|\mathbf{u}\| \leq \omega_G \|\mathbf{y}\| \|\mathbf{u}\| = \omega_G \|\mathbf{x}_1\|.$$

Entropy waves

3. The remaining steps will not increase the norm of $(\mathbf{P}\mathbf{x}_1)^{\parallel}$, and so, by paying ω_G in the final error, we may only consider $\mathbf{x}_2 = (\mathbf{P}\mathbf{x}_1)^{\perp}$. Applying the second H step,

$$\|\tilde{\mathbf{W}}_H \mathbf{x}_2\| \leq \omega_H \|\mathbf{x}_2\|.$$

Hence, $\|\mathbf{W}\mathbf{x}\| \leq \omega_G + \omega_H$.

To recap, we "paid" for two steps ($\deg(G \circled{z} H) = d_2^2$) and "gained" one ω_H (rather than ω_H^2) as the first step was wasted.

Entropy waves

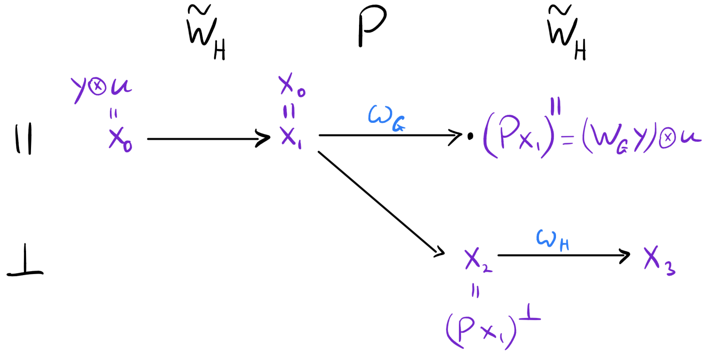

Figure: The Zig-Zag product applied to a "uniform in clouds" ($\parallel$) error vector.

Entropy waves

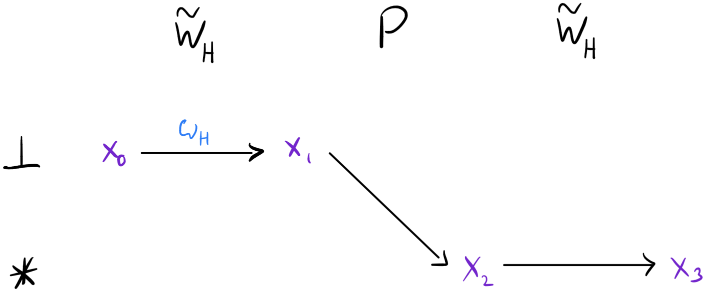

Figure: The Zig-Zag product applied to a “perpendicular in clouds” (⊥) error vector.

Extra space

Overview

Entropy waves

What happens if we take three H steps? $\mathbf{W} = \tilde{\mathbf{W}}_H \mathbf{P} \tilde{\mathbf{W}}_H \mathbf{P} \tilde{\mathbf{W}}_H$

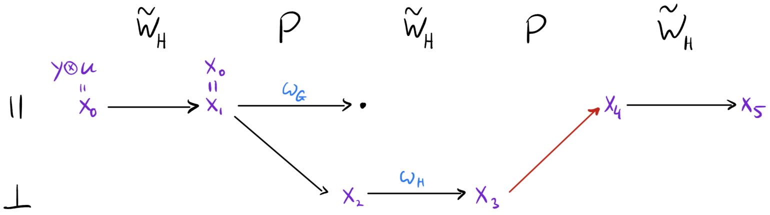

Figure: What goes wrong using 3 H steps. The second application of the **P** operator "flows entropy back" to the clouds, deeming the third $\tilde{\mathbf{W}}_H$ step ineffective.

Extra space

Locally invertible edge rotations

Definition

We say that a rotation map $\pi_G : V \times [d] \to V \times [d]$ is locally invertible if there exists $\phi : [d] \to [d]$ such that

$$\pi_G(g, h) = (\dot{\pi}_G(g, h), \phi(h)).$$

For example, a Cayley graph has a locally invertible edge rotation map. Indeed, if $G = \mathrm{Cay}(\Gamma, S)$ then $\pi_G(g, s) = (g + s, -s)$.

The construction

Ingredients.

- Let G be a d_1-regular graph on n vertices, d_1 a power of two. Specifically, we take G to be the graph obtained using the Zig-Zag product ($\omega_G = O(1/d_1^{1/4})$).
- Let $S \subseteq \mathbb{F}_2^{2\log_2 d_1}$ be an $\varepsilon = \omega_H$-biased set of size

$$|S| = d_2 = O\left(\left(\frac{\log d_1}{\omega_H}\right)^2\right).$$

- Let $H = \mathrm{Cay}(\mathbb{F}_2^{2\log_2 d_1}, S)$.
- Given $h \in H$ we break $h = (A(h), B(h))$ to equal parts, each consisting of $\log d_1$ bits, we dub registers.

The construction

Define the d_2^3-regular graph G_3 on the vertex set $V_G \times V_H$ as follows. $\dot{\pi}_{G_3}((g,h),(a,b,c)) = (g_b, h'_c)$ where

1. $h_a = \dot{\pi}_H(h, a)$
2. $g_a = \dot{\pi}_G(g, A(h_a))$
3. $h'_a = (\phi_G(A(h_a)), B(h_a))$
4. $h_b = \dot{\pi}_H(h'_a, b)$
5. $g_b = \dot{\pi}_G(g_a, B(h_b))$
6. $h'_b = (A(h_b), \phi_G(B(h_b)))$
7. $h_c = \dot{\pi}_H(h'_b, c)$

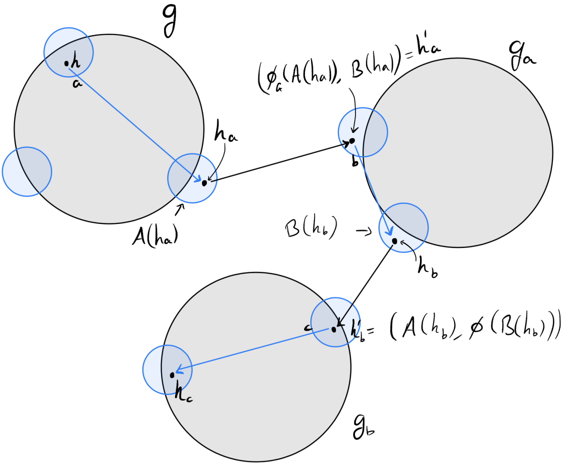

Figure: The width-3 replacement product.

Analysis

Define the operator $\tilde{\mathbf{W}}_H = \mathcal{I} \otimes \mathbf{W}_H$ and the operators $\mathbf{P}_A, \mathbf{P}_B$ that satisfy

$$\mathbf{P}_A(\mathbf{e}(g) \otimes \mathbf{e}(h)) = \mathbf{e}(\dot{\pi}_G(g, A(h))) \otimes \mathbf{e}((\phi_G(A(h)), B(h)))$$
$$\mathbf{P}_B(\mathbf{e}(g) \otimes \mathbf{e}(h)) = \mathbf{e}(\dot{\pi}_G(g, B(h))) \otimes \mathbf{e}((A(h), \phi_G(B(h))))$$

Claim

The random walk matrix of G_3 is given by

$$\mathbf{W} = \tilde{\mathbf{W}}_H \mathbf{P}_B \tilde{\mathbf{W}}_H \mathbf{P}_A \tilde{\mathbf{W}}_H$$

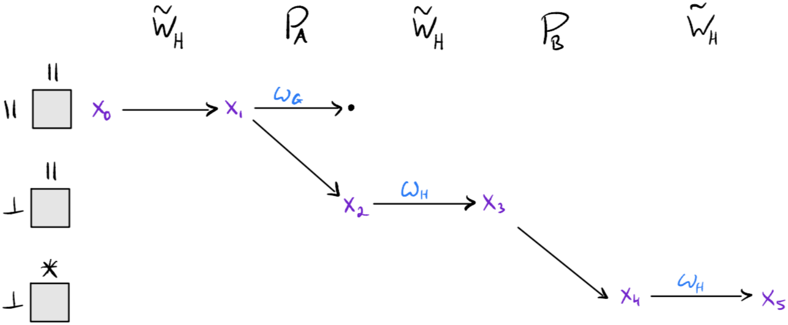

Figure: The width-3 replacement product applied to a $(\|, \|)$ error vector.

Uniform in clouds analysis - first $\tilde{\mathbf{W}}_H$ and $\mathbf{P_A}$ steps

Consider the operator $\mathbf{W}$ applied to a “uniform in clouds” error vector $\mathbf{x} = \mathbf{x}_0 = \mathbf{y} \otimes (\mathbf{u} \otimes \mathbf{u})$, where $\mathbf{y} \perp \mathbf{u}$.

1. First, $\mathbf{x}_1 = \tilde{\mathbf{W}}_H \mathbf{x}_0 = \mathbf{x}_0$. So, as before, the first H step is wasted.
2. Next we have

$$\mathbf{P}_A \mathbf{x}_1 = \mathbf{P}_A(\mathbf{y} \otimes (\mathbf{u} \otimes \mathbf{u})) = (\mathbf{P}(\mathbf{y} \otimes \mathbf{u})) \otimes \mathbf{u}.$$

We decompose $\mathbf{z}_2 = \mathbf{P}(\mathbf{y} \otimes \mathbf{u}) = \mathbf{z}_2^{\|} + \mathbf{z}_2^{\perp}$. As before,

$$\mathbf{z}_2^{\|} = (\mathbf{W}_G \mathbf{y}) \otimes \mathbf{u},$$
$$\mathbf{z}_2^{\perp} = \sum_g \mathbf{e}(g) \otimes \mathbf{z}_{2,g}^{\perp},$$

where $\mathbf{z}_{2,g}^{\perp} \perp \mathbf{u}$ for every g.

Uniform in clouds analysis - first $\tilde{\mathbf{W}}_H$ and $\mathbf{P_A}$ steps

As before,

$$\|\mathbf{z}_2^{\|}\| = \|(\mathbf{W}_G\mathbf{y}) \otimes \mathbf{u}\| = \|(\mathbf{W}_G\mathbf{y})\|\|\mathbf{u}\| \leq \omega_G\|\mathbf{y} \otimes \mathbf{u}\|.$$

Subsequent operations will not increase the norm of $\mathbf{z}_2^{\|}$. Thus, we record a ω_G increase in norm and ignore $\mathbf{z}_2^{\|}$. Namely, we continue the analysis with

$$\mathbf{x}_2 = \mathbf{z}_2^{\perp} \otimes \mathbf{u} = \sum_g \mathbf{e}(g) \otimes (\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u}).$$

Uniform in clouds analysis - second $\tilde{\mathbf{W}}_H$ step

Define

$$\mathbf{x}_3 = \tilde{\mathbf{W}}_H \mathbf{x}_2 = \sum_g \mathbf{e}(g) \otimes \mathbf{W}_H(\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u}).$$

We have that $\|\mathbf{x}_3\| \leq \omega_H \|\mathbf{x}_2\|$. Indeed,

$$\begin{aligned}
\|\mathbf{x}_3\|^2 &= \left\| \sum_g \mathbf{e}(g) \otimes \mathbf{W}_H(\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u}) \right\|^2 \\
&= \sum_g \|\mathbf{W}_H(\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u})\|^2 \\
&\leq \omega_H^2 \sum_g \|\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u}\|^2 \\
&= \omega_H^2 \|\mathbf{x}_2\|^2.
\end{aligned}$$

Uniform in clouds analysis - second $\tilde{\mathbf{W}}_H$ step

As for the structure of $\mathbf{x}_3$, we have that

$$\mathbf{x}_3 = \sum_g \mathbf{e}(g) \otimes \mathbf{W}_H(\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u})$$

is a linear combination of vectors such that within clouds they are of the form $\mathbf{W}_H(\mathbf{z}^{\perp} \otimes \mathbf{u})$ (for $\mathbf{z}^{\perp} \perp \mathbf{u}$).

We introduce a useful notation. For $\mathbf{x} \in \mathbb{R}^n$ and $\pi \in S_n$ we define $\pi\mathbf{x} \in \mathbb{R}^n$ by $(\pi\mathbf{x})(i) = \mathbf{x}(\pi(i))$.

Uniform in clouds analysis - second $\tilde{\mathbf{W}}_H$ step

Recall that $H = \mathrm{Cay}(\mathbb{F}_2^{2\log_2 d_1}, S)$. Thus, for $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{d_1}$ and $v_1, v_2 \in [d_1]$,

$$(\mathbf{W}_H(\mathbf{z}_1 \otimes \mathbf{z}_2))(v_1, v_2) = 2^{-d_2} \sum_{s=(s_1,s_2)\in S} (\mathbf{z}_1 \otimes \mathbf{z}_2)(v_1 + s_1, v_2 + s_2)$$
$$= 2^{-d_2} \sum_{s=(s_1,s_2)\in S} \mathbf{z}_1(v_1 + s_1)\mathbf{z}_2(v_2 + s_2).$$

Hence,

$$\mathbf{W}_H(\mathbf{z}_1 \otimes \mathbf{z}_2) = 2^{-d_2} \sum_{s=(s_1,s_2)\in S} (\pi_{s_1}\mathbf{z}_1) \otimes (\pi_{s_2}\mathbf{z}_2),$$

where π_s is the involution that sends $i \in [2\log_2 d_1]$ to $i + s$. In particular, if $\mathbf{z}_1 \perp \mathbf{u}$ and $\mathbf{z}_2 = \mathbf{u}$ we have that $\mathbf{W}_H(\mathbf{z}_1 \otimes \mathbf{z}_2)$ is a linear combination of vectors with the same structure.

Uniform in clouds analysis - the $\mathbf{P_B}$ step

To recap,

$$\mathbf{x}_3 = \sum_g \mathbf{e}(g) \otimes \mathbf{W}_H(\mathbf{z}_{2,g}^{\perp} \otimes \mathbf{u}) = \sum_g \mathbf{e}(g) \otimes (\mathbf{z}_{3,g}^{\perp} \otimes \mathbf{u}).$$

As for the structure of $\mathbf{P}_B\mathbf{x}_3$, recall that

$$\mathbf{P}_B\mathbf{x}_3 \cong \sum_g (\mathbf{P}(\mathbf{e}(g) \otimes \mathbf{u})) \otimes \mathbf{z}_{3,g}^{\perp},$$

where the isomorphism "switches" the two registers. Thus, $\mathbf{P}_B\mathbf{x}_3$ is in the $(\perp, *)$-space (namely, the first registered remains $\perp$ whereas we lost the structure of the second register.) We set

$$\mathbf{x}_4 = \mathbf{P}_B\mathbf{x}_3 = \sum_g \mathbf{e}(g) \otimes ((\mathbf{z}_{3,g}^{\perp}) \otimes \mathbf{r}_{3,g}),$$

where $\mathbf{r}_{3,g} \in \mathbb{R}^{d_1}$ is arbitrary.

Uniform in clouds analysis - the last $\tilde{\mathbf{W}}_H$ step

Lastly, we have

$$\mathbf{x}_5 = \tilde{\mathbf{W}}_H \mathbf{x}_4 = \sum_g \mathbf{e}(g) \otimes \tilde{\mathbf{W}}_H(\mathbf{z}_{3,g}^{\perp} \otimes \mathbf{r}_{3,g}).$$

Similarly to as was done before, as $\mathbf{z}_{3,g}^{\perp} \otimes \mathbf{r}_{3,g}$ is perpendicular to $\mathbf{u}$ (of dimension d_1^2) it can be shown that $\|\mathbf{x}_5\| \leq \omega_H \|\mathbf{x}_4\|$.

To summarize, on $\mathbf{x} = \mathbf{y} \otimes (\mathbf{u} \otimes \mathbf{u})$, normalized,

$$\|\mathbf{W}\mathbf{x}\| \leq \omega_G + \omega_H^2.$$

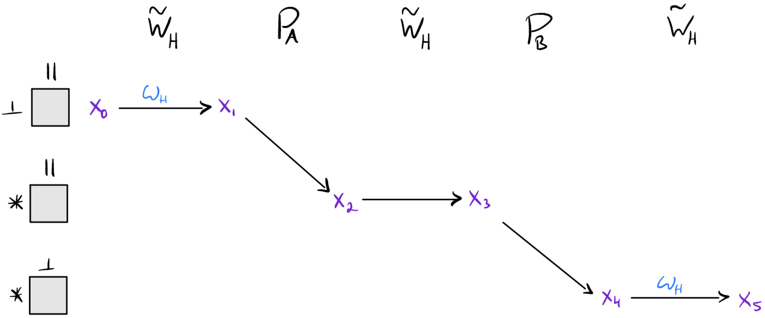

Figure: The width-3 replacement product applied to a $(\perp, \|)$ error vector.

Analysis of the parameters

The degree of G_3 is

$$d = d_2^3 = O\left(\left(\frac{\log d_1}{\omega_H}\right)^6\right).$$

As for the expansion,

$$\omega = \omega(G_3) = O(\omega_H^2 + \omega_G).$$

We take $\omega_G = O(\omega_H^2)$. By the choice of G,

$$d_1 = O(1/\omega_G^4) = O(1/\omega_H^8).$$

Hence,

$$d = \widetilde{O}(1/\omega_H^6) = \widetilde{O}(1/\omega^3).$$

That is, we improved the exponent for 4 to 3.

Analysis of the parameters

By taking G_t to have t H steps.

$$d = \deg(G_t) = d_2^t = O\left(\left(\frac{\log d_1}{\omega_H}\right)^{2t}\right).$$

As for expansion, one can generalize the analysis we did to obtain

$$\omega = \omega(G_t) = O(\omega_H^{t-1} + t\omega_G).$$

We take $\omega_G = O(\omega_H^{t-1}/t)$. As $d_1 = O(1/\omega_G^4) = O(t^4/\omega_H^{4(t-1)})$,

$$d = O\left(\left(\frac{t\log \omega_H}{\omega_H}\right)^{2t}\right).$$

Taking $t = \sqrt{\log(1/\omega)}$, $d = 2^{\widetilde{O}(\sqrt{\log(1/\omega)})} \cdot (1/\omega)^2 = (1/\omega)^{2+o(1)}$.