Riemann's Theorem

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Discussion

Let X/K be a nonsingular complete curve with function field $K(X)/K$. Let $P_1, \ldots, P_s$ be a set of points on X. We will be interested in questions such as

- *Is it possible to find a nonconstant function $\alpha \in K(X)$ that has no poles outside this set?* Yes.
- *Given $a \in \mathbb{N}$ is there a function $\alpha \in K(X)$ that has pole order precisely a at P ($v_P(\alpha) = -a$) and no other poles?* True for a "large enough" a.
- *How many zeros and poles a given function $\alpha \in K(X) \setminus K$ has?* Same number - $[K(X) : K(\alpha)]$!

Example

Let K be algebraically closed. Let $\mathbb{P}^1/K$ be the projective line with field of functions $K(x)$. Recall that we can identify $\mathbb{P}^1$ with $K \cup \{\infty\}$ in such a way that $K[x]$ is the ring of functions defined everywhere on K.

Let $\alpha_1, \ldots, \alpha_s \in K$ and $a_\infty, a_1, \ldots, a_s \in \mathbb{N}$. Take any $\beta \notin \{\alpha_1, \ldots, \alpha_s\}$ and set $b = a_\infty + a_1 + \cdots + a_s$. Then the function

$$\frac{(x-\beta)^b}{(x-\alpha_1)^{a_1} \cdots (x-\alpha_s)^{a_s}} \in K(x)$$

has

- pole order a_∞ at ∞.
- pole order a_i at α_i for $i = 1, \ldots, s$.
- no other poles.

The following proposition that we will prove in the next course asserts that every nonzero function in L has a finite number of poles and zeros.

Proposition

Let L/K be a function field. Then, for every $\alpha \in L^\times$

$$|\{v \in \mathcal{V}(L/K) \mid v(\alpha) \neq 0\}| < \infty.$$

Definition

Let X/K be a nonsingular complete curve with function field L/K. The group of divisors of L/K is the free abelian group generated by $\{x_v \mid v \in \mathcal{V}(L/K)\}$. That is,

$$\mathrm{Div}(L/K) = \bigoplus_{v \in \mathcal{V}(L/K)} \mathbb{Z}x_v.$$

Identifying $\mathcal{V}(L/K)$ with X as usual, we can write $\mathrm{Div}(L/K)$ as

$$\mathrm{Div}(X) = \bigoplus_{P \in X} \mathbb{Z}P.$$

An element $D \in \mathrm{Div}(X)$ is called a divisor of L. $D = \sum_P a_P P$ where $a_P = 0$ except for finitely many $P \in X$.

Definition

An element in L (namely, a function) is naturally mapped to a divisor. Given $\alpha \in L$ we define the divisor

$$(\alpha) = \sum_{P \in X} v_P(\alpha) P.$$

The previous proposition guarantees that (α) is well-defined.

Definition

Let X/K be a nonsingular complete curve. Let $P \in X$ and v the corresponding valuation. Let $K_v = \mathcal{O}_v/\mathcal{M}_v$ be the residue class field. Recall that $K \hookrightarrow K_v$. We define

$$\deg(P) = [K_v : K].$$

When $\deg(P) = 1$ we say that P is a rational point of X.

We will prove the following claim in the next course.

Claim

$\deg(P)$ *is always finite. In particular, if K is algebraically closed then all points on X are rational.*

Definition

Let X/K be a nonsingular complete curve. Let $D = \sum_P a_P P \in \mathrm{Div}(X)$. We define

$$\deg(D) = \sum_P a_P \deg(P).$$

Theorem

Let X/K be a nonsingular complete curve. Then, $\forall \alpha \in K(X)^\times$,

$$\deg((\alpha)) = \sum_P v_P(\alpha) \deg(P) = 0.$$

Moreover,

$$\sum_P \max(0, v_P(\alpha)) \deg(P) = [L : K(\alpha)].$$

Discussion

The above theorem proves that every function has the same number of zeros and poles. Moreover, this number is the degree of the extension $L/K(\alpha)$.

Proving this theorem requires a fair amount of work and, in particular, makes use of the fundamental equality which we'll prove in the next course when we will discuss factorization in ring extensions.

Definition

1. We denote the zero divisor ($a_P = 0$ for all $P \in X$) by 0.
2. A divisor $D = \sum_P a_P P$ is called positive if $a_P \geq 0$ for all $P \in X$.

Definition

We put a partial ordering $\geq$ on $\mathrm{Div}(X)$ in the natural way:
$D' \geq D \iff D' - D$ is a positive divisor (namely, $D' - D \geq 0$).

Definition (Riemann-Roch spaces)

Let L/K be a function field. For $D \in \mathrm{Div}(L/K)$ define the Riemann-Roch space of D by

$$\mathcal{L}(D) = \left\{\alpha \in L^{\times} \mid (\alpha) + D \geq 0\right\} \cup \{0\}.$$

Example

If $P, Q \in X$ and $D = 3P - 2Q$ then $\alpha \in \mathcal{L}(D)$ if and only if

- $v_P(\alpha) \geq -3$. Read α is allowed to have at most "3 poles" at P.
- α must not have poles anywhere other than P.
- $v_Q(\alpha) \geq 2$. Read α must have at least "2 zeros" at Q.

Claim

Take $L = K(x)$. Then,

$$\mathcal{L}(rP_\infty)$$

is precisely the set of all polynomials in x with degree at most r.

Proof.

By definition $\alpha = \frac{f}{g} \in \mathcal{L}(rP_\infty)$ if and only if

- $v_\infty(\alpha) \geq -r$.
- α has no other pole.

The second item implies that $g = 1$. Otherwise, α would have a pole at a "point" which corresponds to an irreducible component of $g \implies \alpha = f$. Since $v_\infty(\alpha) = -\deg(f)$ the first item implies that $\deg(f) \leq r$. □

Claim

$\mathcal{L}(0) = K$.

Proof.

$\alpha \in \mathcal{L}(0) \iff \alpha$ has no poles. The proof then follows since $\mathcal{O}_X(X) = K$. □

Claim

$\mathcal{L}(D)$ is a K-vector space.

Proof.

Readily follows from the fact that for every $P \in X$

$$v_P(\alpha + \beta) \geq \min(v_P(\alpha), v_P(\beta))$$

and since $v_P(K^\times) = 0$. □

Definition

We denote $\ell(D) = \dim_K(\mathcal{L}(D))$.

Claim

$$\deg(D) < 0 \quad \Longrightarrow \quad \ell(D) = 0$$

Proof.

Let $0 \neq \alpha \in \mathcal{L}(D)$. Then $(\alpha) + D \geq 0$. Thus

$$\deg((\alpha) + D) \geq 0.$$

But

$$\deg((\alpha) + D) = \deg((\alpha)) + \deg(D) = \deg(D).$$

To conclude, we showed that $\ell(D) > 0 \implies \deg(D) \geq 0$. □

Proposition

For every divisor $D \geq 0$

$$\ell(D) \leq \deg(D) + 1.$$

Remark

For simplicity, we will prove the proposition for an algebraically closed field. This is mostly for ease of notation. The assumption will be used as follows: if v *is a valuation and* $K_v = \mathcal{O}_v / \mathcal{M}_v$ *is the corresponding residue field then, as we saw,* $K_v = K$.

For the proof we need to define Laurent expansions.

Lemma

Let L/K be a function field with K algebraically closed. Let $v \in \mathcal{V}(L/K)$. Let $\mathcal{O}_v, \mathcal{M}_v$ be the associated local ring and maximal ideal, and $K_v = \mathcal{O}_v/\mathcal{M}_v \cong K$ the residue field. Let π be a generator of $\mathcal{M}_v$.

For every $\beta \in L^\times$ there exist:

- *a unique sequence $\{b_i\}_{i \geq v(\beta)}^{\infty}$ with $b_i \in K$, and*
- *a unique sequence $\{\beta_j\}_{j \geq v(\beta)+1}^{\infty}$ with $\beta_j \in L^\times$ and $v(\beta_j) \geq j$*

such that $\forall n \geq v(\beta)$

$$\beta = \sum_{i=v(\beta)}^{n} b_i \pi^i + \beta_{n+1}.$$

Proof.

Let P the point corresponding to v. First, consider $\alpha \in \mathcal{O}_v^\times$. Define

$$a_0 = \alpha(P) \in K$$
$$\alpha_1 = \alpha - a_0 \in L$$

Note that

$$\alpha_1(P) = (\alpha - a_0)(P) = \alpha(P) - a_0(P) = 0.$$

Thus

$$\alpha_1 \in \mathcal{M}_v \implies v(\alpha_1) \geq 1 \implies v\left(\frac{\alpha_1}{\pi}\right) \geq 0 \implies \frac{\alpha_1}{\pi} \in \mathcal{O}_v.$$

This gives the expansion for $n = 0$: $\alpha = a_0\pi^0 + \alpha_1$. $\square$

Proof.

We now repeat for $\frac{\alpha_1}{\pi}$:

$$a_1 = \left(\frac{\alpha_1}{\pi}\right)(P)$$
$$\alpha_2 = \frac{\alpha_1}{\pi} - a_1 = \frac{\alpha - a_0}{\pi} - a_1.$$

We thus get expansion for $n = 1$

$$\alpha = a_0 + a_1\pi + \alpha_2\pi.$$

Indeed, $\alpha_2(P) = 0 \implies v(\alpha_2) \geq 1 \implies v(\alpha_2\pi) \geq 2$.

For a general $0 \neq \beta \in L$ recall that we can write $\beta = \pi^{v(\beta)}\alpha$ where $\alpha \in \mathcal{O}_v^\times$. Expand α and plug in. Uniqueness is left as an exercise. $\square$

Recall, we wish to prove

Proposition

For every divisor $D \geq 0$

$$\ell(D) \leq \deg(D) + 1.$$

Proof.

Write $D = \sum_{i=1}^{s} a_{P_i} P_i$ with $a_{P_i} > 0$. Let π_i be a generator for $\mathcal{M}_{P_i}$. Laurent expand α at P_i to get

$$\alpha = \sum_{j=v_{P_i}(\alpha)}^{-1} a_{i,j}\pi_i^j + \alpha_{i,0}$$

Recall that $v_{P_i}(\alpha_{i,0}) \geq 0$. □

Proof.

Define the map

$$\mu : \mathcal{L}(D) \to K^{a_{P_1}} \times \cdots \times K^{a_{P_s}}$$
$$\alpha \mapsto (a_{1,v_{P_1}(\alpha)}, \ldots, a_{1,-1}, \ldots, a_{s,v_{P_s}(\alpha)}, \ldots, a_{s,-1}).$$

Observe that μ is a homomorphism of K-vector spaces and so

$$\ell(D) = \dim \ker(\mu) + \dim \mathrm{Im}(\mu).$$

Clearly, $\dim \mathrm{Im}(\mu) \leq \sum_{i=1}^{s} a_{P_i} = \deg(D)$ (note that we use that all points are rational).

Now, if $\alpha \in \ker(\mu)$ then $\alpha \in \mathcal{O}_{P_i}$ for all $i \in [s]$. But together with $\alpha \in \mathcal{L}(D)$ we get that α has no poles, and so $\alpha \in K$. Thus, $\ker(\mu) = K$ and in particular $\dim \ker(\mu) = 1$. □

In fact more is true

Proposition

Let X/K be a nonsingular complete curve. Then,

$$\ell(D) \leq \max(0, \deg(D) + 1).$$

Theorem (Riemann's Theorem)

Let X/K be a nonsingular complete curve. Then, there exists $g = g(X) \in \mathbb{N}$ such that for every $D \in \mathrm{Div}(X)$

$$\ell(D) \geq \deg(D) + 1 - g.$$

g is called the genus of X.

This is a **BIG** theorem. We will prove it in the next course. In fact, we will prove a strengthening called the Riemann-Roch Theorem.