Random Walks on Graphs

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Random walks

Let $G = (V, E)$ be an undirected d-regular graph, and $\mathbf{p}$ a probability distribution on V, thought of as a vector $\mathbf{p} \in \mathbb{R}^V$.

A random step on G, starting from a probability distribution $\mathbf{p}$, is the process in which we

1. sample v according to $\mathbf{p}$;
2. sample a neighbor u of v uniformly at random, and return u.

If $\mathbf{p}_{\mathrm{new}}$ is the distribution over V after taking a random step, then for every $v \in V$,

$$\mathbf{p}_{\mathrm{new}}(v) = \frac{1}{d} \sum_{u \in \Gamma(v)} \mathbf{p}(u).$$

Random walks

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Note that

$$\mathbf{p}_{\text{new}} = \frac{1}{d} \mathbf{M}_G \mathbf{p} = \mathbf{W}_G \mathbf{p}.$$

A length t random walk is the probabilistic process of taking t consecutive random steps. The corresponding distributions are given by

$$\mathbf{p}_t = \mathbf{W} \mathbf{p}_{t-1} = \mathbf{W}^2 p_{t-2} = \cdots = \mathbf{W}^t \mathbf{p}_0 = \mathbf{W}^t \mathbf{p}.$$

The normalized adjacency matrix

We denote the eigenvalues of **W** by $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$.

One can show that $\omega_1 \leq 1$. The Perron-Frobenius Theorem implies that

$$\mathrm{Spec}(\mathbf{W}) \subset [-1, 1].$$

We denote

$$\omega(G) = \max(|\omega_2|, |\omega_n|).$$

By P-F, G is connected and not bipartite $\iff \omega(G) < 1$.

Theorem

Assume that G is connected and not bipartite. Then, a random walk from any initial distribution converges to the stable distribution π which is uniform over V.

The rate of convergence

Theorem

Let $\mathbf{p}_0 = \delta(u)$ for some $u \in V$. Then, for every $v \in V$,

$$|\mathbf{p}_t(v) - \boldsymbol{\pi}(v)| \leq \omega(G)^t.$$

So, we see that the smaller $\omega(G) = \max(|\omega_2|, |\omega_n|)$ is, the faster is the convergence.

In the context of Cheeger's inequality, we asked how small can ω_2 be. Here we see a stronger bound is required.