Noetherianity in Separable Extensions

Introduction to Algebraic-Geometric Codes. Fall 2019

April 30, 2019

Setting (The AKLB setting)

Let A be an integrally closed domain with $K = \mathrm{Frac}(A)$. Let L/K be a separable extension of degree $[L : K] = n$. Let B be the integral closure of A in L.

Claim

In the AKLB setting, there exists $\beta \in B$ such that $L = K(\beta)$.

We recall the following theorem from field theory.

Theorem

Every finite separable extension is simple.

Proof of Claim.

L/K is finite + separable $\implies \exists \gamma \in L$ s.t. $L = K(\gamma)$.
Recall that "$L = B/A$" $\implies \exists \alpha \in A, \beta \in B\ \gamma = \beta/\alpha$.
$\implies L = K(\gamma) = K(\beta/\alpha) = K(\beta)$. □

Another theorem we recall here without a proof.

Theorem

Let L/K be an algebraic extension. Let S be all elements in L that are separable over K. Then, S is a field.

Proposition

Assume the AKLB setting. Write $L = K(\beta)$. *Then,* $\exists d \in A \setminus \{0\}$ *s.t. the* A*-module* B *is contained in*

$$F = A\frac{\beta^0}{d} + A\frac{\beta}{d} + \cdots + A\frac{\beta^{n-1}}{d}.$$

Corollary (Main message from this unit!)

Assume AKLB. Then, A *is noetheiran* $\implies$ B *is a f.g.* A*-module. In particular,* B *is a noetheiran ring.*

Proof of Corollary.

A noetheiran + F f.g. A-module $\implies$ F noetherian A-module.
B an A-submodule of F $\implies$ B f.g. A-module. □

Proof of Proposition.

Since $L = K(\beta)$, $\forall b \in B\ \exists x_0, \ldots, x_{n-1} \in K$ s.t. $b = \sum_{i=0}^{n-1} x_i \beta^i$.
L/K separable of degree $n \implies \Gamma_{L/K} = \{\sigma_1, \ldots, \sigma_n : L \hookrightarrow \bar{K}\}$.
Define the $n \times n$ matrix M by $M_{i,j} = \sigma_i(\beta^{j-1})$.Observe that

$$\begin{pmatrix} \sigma_1(b) \\ \vdots \\ \sigma_n(b) \end{pmatrix} = M \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

Indeed,

$$\sigma_i(b) = \sigma_i\left(\sum_{j=0}^{n-1} x_j \beta^j\right) = \sum_{j=0}^{n-1} x_j \sigma_i(\beta^j).$$

□

Proof of Proposition (cont.)

$$\begin{pmatrix} \sigma_1(b) \\ \vdots \\ \sigma_n(b) \end{pmatrix} = M \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$\implies M^* \begin{pmatrix} \sigma_1(b) \\ \vdots \\ \sigma_n(b) \end{pmatrix} = M^* M \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \det(M) \cdot x_0 \\ \vdots \\ \det(M) \cdot x_{n-1} \end{pmatrix}$$

□

Proof of Proposition (cont.)

$$M^* \begin{pmatrix} \sigma_1(b) \\ \vdots \\ \sigma_n(b) \end{pmatrix} = \begin{pmatrix} \det(M) \cdot x_0 \\ \vdots \\ \det(M) \cdot x_{n-1} \end{pmatrix}$$

M^* entries are integral over A $+$ $\sigma_i(b)$ integral over A
$\implies$ $\det(M) \cdot x_i$ are all integral over A.

Observe that if $\det(M) \in K^\times$ we are done! Indeed, in such case,

$$b = \sum_{i=0}^{n-1} x_i \beta^i = \frac{1}{\det(M)} \cdot \sum_{i=0}^{n-1} (\det(M) \cdot x_i) \beta^i.$$

$\det(M) \in K^\times$ + $\det(M)$ integral over A $\implies$ $\det(M) \in A$.
Similarly, $\det(M) \cdot x_i$ are all in A. So taking $d \triangleq \det(M)$ we would be done. However, $\det(M)$ may not be in $K^\times$. □

Claim

$\det(M)^2 \in K$.

Proof.

For simplicity, we are going to assume that $L \subseteq \bar{K}$. Using Steinitz's theorems one can handle the general case (try it!)
Take any $\nu \in \Gamma_K$. $\nu : \bar{K} \hookrightarrow \bar{K}$ an automorphism that fixes K.
Observe that

$$\{\nu \circ \sigma_1, \ldots, \nu \circ \sigma_n\} = \{\sigma_1, \ldots, \sigma_n\}.$$

Define $\nu \circ M$ by $(\nu \circ M)_{i,j} = \nu(M_{i,j})$. By the above, $\nu \circ M$ is M up to row permutation $\implies \det(\nu \circ M) = \pm \det(M)$.
But $\det(\nu \circ M) = \nu(\det(M))$.
So, $\nu(\det(M)) = \pm \det(M) \quad \implies \quad \nu(\det(M)^2) = \det(M)^2$.
The proof follows since $\det(M)^2$ is separable over K. □

Corollary

$\det(M)^2 \in A$.

Proof.

$\det(M)^2 \in K \quad + \quad \det(M)$ is integral over A. The proof follows since A is integrally closed. □

Claim

$\det(M)^2 \cdot x_i \in A$.

Proof.

$\det(M)^2 \cdot x_i \in K$ as $\det(M)^2 \in K$ and $x_i \in K$. Now,

$$\det(M)^2 \cdot x_i = \det(M) \cdot (\det(M) \cdot x_i)$$

and we proved that $\det(M)$ and $\det(M) \cdot x_i$ are integral over A. The proof follows as A is integrally closed. □

Proof of Proposition (cont.)

So we can write

$$b = \sum_{i=0}^{n-1} x_i \beta^i$$
$$= \frac{1}{\det(M)^2} \cdot \sum_{i=0}^{n-1} (\det(M)^2 \cdot x_i) \beta^i.$$

Observe that $\det(M)^2 \in A$ and $\det(M)^2 \cdot x_i \in A$ as stated.
Only thing left is to show that $\det(M)^2 \neq 0$ (check!) □

To summarize this unit,

Theorem

Let A be an integrally closed domain with $K = \mathrm{Frac}(A)$. Let L/K be a finite and separable extension. Let B be the integral closure of A in L. Then,

$$A \text{ noetherian} \implies B \text{ noetherian}.$$