Hurwitz Genus Formula

Unit 22

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Overview

Hurwitz Genus Formula

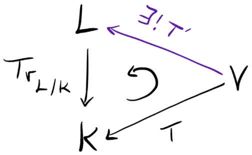
Throughout this unit F/L is a finite separable extension of E/K.

Lemma 1

Let L/K *be a finite separable field extension. Let* V *be an* L*-vector space (and so* V *is also a* K*-vector space). Let* $T : V \to \mathrm{K}$ *be a* K*-linear map.*

Then, $\exists! T' : V \to \mathrm{L}$ *that is* L*-linear s.t.*

$$\mathrm{Tr}_{\mathrm{L/K}} \circ T' = T.$$



We omit the proof of this fact (see Dan Haran's lecture notes; Chapter 33).

Adeles - recall

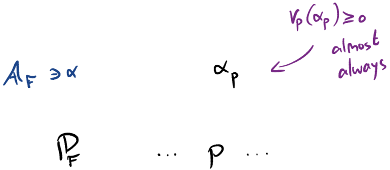
Recall that an adele of F/L is a function $\alpha : \mathbb{P}(F/L) \to F$ that maps $\mathfrak{P} \to \alpha_{\mathfrak{P}}$ s.t. $v_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \geq 0$ almost always.

The set of adeles of F/L is denoted by $\mathbb{A}_{F/L}$ or $\mathbb{A}_F$. Recall that $\mathbb{A}_F$ is an F-algebra. Multiplying by elements of F is done via the embedding $F \hookrightarrow \mathbb{A}_F$ where $x \mapsto [x]$ in which $[x]_{\mathfrak{P}} = x$.

For $\mathfrak{a} \in \mathcal{D}(F/L)$ we defined

$$\Lambda_F(\mathfrak{a}) = \{\alpha \in \mathbb{A}_F \ \mid \ \forall \mathfrak{P} \in \mathbb{P}(F/L) \quad v_{\mathfrak{P}}(\alpha) + v_{\mathfrak{P}}(\mathfrak{a}) \geq 0\}.$$

We sometimes write $\Lambda(\mathfrak{a})$ for short.



Adeles of extensions

Definition 2

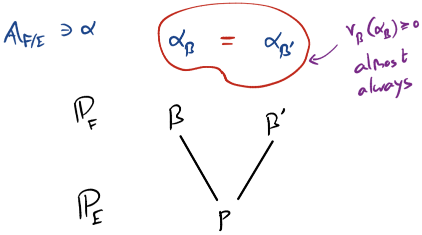
We extend the above definition to extensions.

$$\mathbb{A}_{F/E} = \{\alpha \in \mathbb{A}_F \mid \mathfrak{P}_1 \cap E = \mathfrak{P}_2 \cap E \implies \alpha_{\mathfrak{P}_1} = \alpha_{\mathfrak{P}_2}\}.$$

Note that $F \hookrightarrow \mathbb{A}_{F/E} \subseteq \mathbb{A}_F$ and so $\mathbb{A}_{F/E}$ is an F-subalgebra of $\mathbb{A}_F$.

Moreover, for $\mathfrak{a} \in \mathcal{D}(F/L)$ we define

$$\Lambda_{F/E}(\mathfrak{a}) = \mathbb{A}_{F/E} \cap \Lambda_F(\mathfrak{a}).$$



Adeles of extensions

Definition 3

We extend $\mathrm{Tr}_{F/E} : F \to E$ to the map

$$\mathrm{Tr}_{F/E} : \mathbb{A}_{F/E} \to \mathbb{A}_E$$

as follows: For $\alpha \in \mathbb{A}_{F/E}$ and $\mathfrak{p} \in \mathbb{P}(E)$,

$$(\mathrm{Tr}_{F/E}(\alpha))_{\mathfrak{p}} = \mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}})$$

where $\mathfrak{P}$ is some prime divisor lying over $\mathfrak{p}$.

We need to prove that indeed

$$\mathrm{Tr}_{F/E}(\alpha) \in \mathbb{A}(E).$$

Namely, we need to show that $\mathrm{Tr}_{F/E}(\alpha)_{\mathfrak{p}} \geq 0$ almost always.

Adeles of extensions

We need to show that $\mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha)_{\mathfrak{p}} \geq 0$ almost always.

As $\alpha \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ we have that $\alpha_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}$ almost always. Thus, for almost all $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$,

$$\forall \mathfrak{P}/\mathfrak{p} \quad \alpha_{\mathfrak{P}} \in \bigcap_{\mathfrak{P}'/\mathfrak{p}} \mathcal{O}_{\mathfrak{P}'} = \mathcal{O}'_{\mathfrak{p}}.$$

Recall that $\mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\mathcal{O}'_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{p}}$, and so for almost all $\mathfrak{p}$,

$$(\mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha))_{\mathfrak{p}} = \mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}}) \in \mathcal{O}_{\mathfrak{p}},$$

thus establishing that $\mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha) \in \mathbb{A}(\mathsf{E})$.

Adeles of extensions

We further remark that

$$\mathrm{Tr}_{F/E}([x]) = [\mathrm{Tr}_{F/E}(x)].$$

Lemma 4

For every $\mathfrak{a} \in \mathcal{D}(F)$ *we have that*

$$\mathbb{A}_F = \mathbb{A}_{F/E} + \Lambda_F(\mathfrak{a}).$$

Proof.

The inclusion $\mathbb{A}_F \supset \mathbb{A}_{F/E} + \Lambda_F(\mathfrak{a})$ is obvious. For the other inclusion, take $\alpha \in \mathbb{A}_F$. We first construct some $\beta \in \mathbb{A}_{F/E}$ as follows.

Proof.

Take $\mathfrak{p} \in \mathbb{P}(E)$. The set of $\mathfrak{P}/\mathfrak{p}$ is finite and so by WAT, $\exists x_{\mathfrak{p}} \in F$ s.t.

$$\forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(x_{\mathfrak{p}} - \alpha_{\mathfrak{P}}) \geq -v_{\mathfrak{P}}(\mathfrak{a}).$$

Note that for almost all $\mathfrak{p} \in \mathbb{P}(E)$ we have that $\forall \mathfrak{P}/\mathfrak{p}$ $v_{\mathfrak{P}}(\mathfrak{a}) = 0$.

Moreover, since $\alpha \in \mathbb{A}_F$, for almost all $\mathfrak{P} \in \mathbb{P}(F)$, $v_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \geq 0$. Thus, for almost all $\mathfrak{p}$,

$$\begin{aligned} \forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(x_{\mathfrak{p}}) &= v_{\mathfrak{P}}(x_{\mathfrak{p}} - \alpha_{\mathfrak{P}} + \alpha_{\mathfrak{P}}) \\ &\geq \min(v_{\mathfrak{P}}(x_{\mathfrak{p}} - \alpha_{\mathfrak{P}}), v_{\mathfrak{P}}(\alpha_{\mathfrak{P}})) \\ &\geq 0. \end{aligned}$$

Adeles of extensions

Proof.

With this, we define $\beta : \mathbb{P}(F) \to F$ by

$$\beta_{\mathfrak{P}} = x_{\mathfrak{p}},$$

where $\mathfrak{p} \in \mathbb{P}(E)$ is the prime divisor lying under $\mathfrak{P}$.

$\beta \in \mathbb{A}_F$ as $v_{\mathfrak{P}}(\beta_{\mathfrak{P}}) = v_{\mathfrak{P}}(x_{\mathfrak{p}}) \geq 0$ almost always. Moreover, $\beta \in \mathbb{A}(F/E)$ since we take $\beta_{\mathfrak{P}} = x_{\mathfrak{p}} = \beta_{\mathfrak{P}'}$ for all places $\mathfrak{P}, \mathfrak{P}'$ lying over $\mathfrak{p}$.

Lastly, note that $\alpha - \beta \in \Lambda_F(\mathfrak{a})$. Indeed, $\forall \mathfrak{P} \in \mathbb{P}(F)$,

$$v_{\mathfrak{P}}(\alpha - \beta) = v_{\mathfrak{P}}(\alpha_{\mathfrak{P}} - \beta_{\mathfrak{P}}) = v_{\mathfrak{P}}(\alpha_{\mathfrak{P}} - x_{\mathfrak{p}}) \geq -v_{\mathfrak{P}}(\mathfrak{a}).$$

Thus,

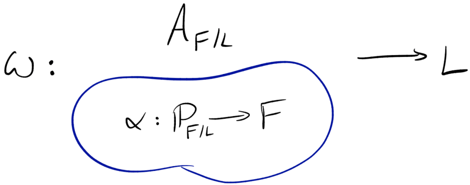
$$\alpha = \beta + (\alpha - \beta) \in \mathbb{A}_{F/E} + \Lambda_F(\mathfrak{a}),$$

concluding the proof. □

Overview

Differential - recall

Recall that a differential of F/L is an L-linear map $\omega : \mathbb{A}_F \to L$ that is nullified on a subspace of the form $\Lambda(\mathfrak{a}) + F$ for some divisor $\mathfrak{a}$.



For a differential $\omega \neq 0$ we defined the canonical divisor

$$(\omega) = \max\left\{\mathfrak{a} \in \mathcal{D}(F) \,:\, \omega|_{\Lambda(\mathfrak{a})+F} = 0\right\}.$$

In particular, $\omega|_{\Lambda((\omega))} = 0$.

Differentials in extensions

Lemma 5

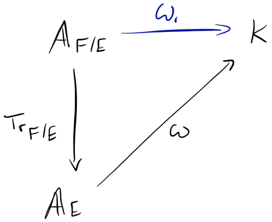
Let $\omega : \mathbb{A}_E \to K$ *be a differential of* E/K*. We define a map*

$$\omega_1 : \mathbb{A}_{F/E} \to K$$

by $\omega_1 = \omega \circ \mathrm{Tr}_{F/E}$*. Then,*

1. ω_1 *is* K*-linear; and*
2. ω_1 *is nullified on* $\Lambda_{F/E}(\mathfrak{a}) + F$*, where*

$$\mathfrak{a} = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$



Differentials in extensions

Proof.

The first item follows since both $\mathrm{Tr}_{F/E}$ and ω are K-linear maps.

For the second item, first note that $\omega_1|_F = 0$. Indeed, $\mathrm{Tr}_{F/E}(F) = E$, and $\omega|_E = 0$.

We turn to prove that $(\omega_1)|_{\Lambda_{F/E}(\mathfrak{a})} = 0$.

Take $\alpha \in \Lambda_{F/E}(\mathfrak{a})$. We need to show that $\omega_1(\alpha) = \omega(\mathrm{Tr}_{F/E}(\alpha)) = 0$. To this end we show that

$$\mathrm{Tr}_{F/E}(\alpha) \in \Lambda_E((\omega)).$$

Equivalently,

$$\forall \mathfrak{p} \in \mathbb{P}(E) \quad v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\alpha)) + v_{\mathfrak{p}}((\omega)) \geq 0.$$

Thus, we need to show that for all $\mathfrak{p}$ and $\mathfrak{P}/\mathfrak{p}$,

$$v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}})) + v_{\mathfrak{p}}((\omega)) \geq 0.$$

Differentials in extensions

Proof.

We want to show that

$$\forall \mathfrak{p}, \mathfrak{P}/\mathfrak{p} \quad \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}})) + \upsilon_{\mathfrak{p}}((\omega)) \geq 0.$$

Fix $\mathfrak{p}$ and let $x \in E$ be s.t. $\upsilon_{\mathfrak{p}}(x) = \upsilon_{\mathfrak{p}}((\omega))$. Then, for all $\mathfrak{P}/\mathfrak{p}$,

$$\begin{aligned}
\upsilon_{\mathfrak{P}}(x\alpha_{\mathfrak{P}}) &= \upsilon_{\mathfrak{P}}(x) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&= e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&= e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&\geq e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}((\omega)) - \upsilon_{\mathfrak{P}}(\mathfrak{a}) \\
&= \upsilon_{\mathfrak{P}}(\mathrm{Con}_{F/E}((\omega)) - \mathfrak{a}) \\
&= \upsilon_{\mathfrak{P}}(-\mathrm{Diff}(F/E)) \\
&= -d(\mathfrak{P}/\mathfrak{p}).
\end{aligned}$$

Thus, $x\alpha_{\mathfrak{P}} \in C_{\mathfrak{p}}$.

Proof.

Fix $\mathfrak{p}$ and let $x \in \mathsf{E}$ be s.t. $\upsilon_{\mathfrak{p}}(x) = \upsilon_{\mathfrak{p}}((\omega))$. Then, $x\alpha_{\mathfrak{P}} \in \mathsf{C}_{\mathfrak{p}}$. Thus,

$$\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x\alpha_{\mathfrak{P}})) \geq 0.$$

Since $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}$ is E-linear, we get that

$$\begin{aligned}\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x\alpha_{\mathfrak{P}})) &= \upsilon_{\mathfrak{p}}(x\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \\ &= \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \\ &= \upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})).\end{aligned}$$

Thus,

$$\upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \geq 0$$

which, recall, concludes the proof. □

Differentials in extensions

Let $\omega : \mathbb{A}_E \to K$ be a differential of E/K. Recall that we have defined the map

$$\omega_1 : \mathbb{A}_{F/E} \to K$$

by $\omega_1 = \omega \circ \mathrm{Tr}_{F/E}$. We further denoted

$$\mathfrak{a} = \mathrm{Con}_{F/E}(\omega) + \mathrm{Diff}(F/E).$$

and proved that ω_1 is nullified on $F + \Lambda_{F/E}(\mathfrak{a})$.

Lemma 6

$\forall \mathfrak{a}' \in \mathcal{D}(F)$

$$\mathfrak{a}' \not\leq \mathfrak{a} \quad \Longrightarrow \quad (\omega_1)|_{\Lambda_{F/E}(\mathfrak{a}')} \neq 0.$$

Differentials in extensions

Proof. (Proof of Lemma 6)

We wish to prove that

$$\forall \mathfrak{a}' \in \mathcal{D}(\mathsf{F}) \ \text{ s.t. } \ \mathfrak{a}' \not\leq \mathfrak{a} \quad \exists \beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(\mathfrak{a}') \ \text{ s.t. } \ \omega_1(\beta) \neq 0.$$

Fix $\mathfrak{a}' \not\leq \mathfrak{a}$ and let $\mathfrak{P}' \in \mathbb{P}(\mathsf{F})$ s.t.

$$\upsilon_{\mathfrak{P}'}(\mathfrak{a}') > \upsilon_{\mathfrak{P}'}(\mathfrak{a}) = \upsilon_{\mathfrak{P}'}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega)) + d(\mathfrak{P}'/\mathfrak{p}),$$

where $\mathfrak{p}$ is the prime divisor lying under $\mathfrak{P}'$. That is,

$$\upsilon_{\mathfrak{P}'}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - \mathfrak{a}') < -d(\mathfrak{P}'/\mathfrak{p}).$$

Define

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \quad \upsilon_{\mathfrak{P}}(z) \geq \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - \mathfrak{a}')\}.$$

J is closed under addition and under multiplication by $\mathcal{O}'_{\mathfrak{p}}$ and so J is an $\mathcal{O}'_{\mathfrak{p}}$-module. Furthermore, $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J)$ is an $\mathcal{O}_{\mathfrak{p}}$-module.

Differentials in extensions

Proof.

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \;\; \upsilon_{\mathfrak{P}}(z) \geq \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - \mathfrak{a}')\}.$$

By WAT $\exists z' \in \mathsf{F}$ s.t.

$$\forall \mathfrak{P}/\mathfrak{p} \quad \upsilon_{\mathfrak{P}}(z) = \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - \mathfrak{a}').$$

In particular, $z' \in J$ and

$$\upsilon_{\mathfrak{P}'}(z') < -d(\mathfrak{P}'/\mathfrak{p}),$$

and so $z' \notin \mathsf{C}_{\mathfrak{p}}$. Thus, $\exists v \in \mathcal{O}'_{\mathfrak{p}}$ s.t.

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(vz') \notin \mathcal{O}_{\mathfrak{p}}.$$

As J is an $\mathcal{O}'_{\mathfrak{p}}$-module, $vz' \in J$ and so $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) \not\subseteq \mathcal{O}_{\mathfrak{p}}$.

Differentials in extensions

Proof.

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \;\; v_{\mathfrak{P}}(z) \geq v_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - \mathfrak{a}')\}.$$

Let $t \in \mathsf{E}$ be with $v_{\mathfrak{p}}(t) = 1$. Thus, for a sufficiently large r,

$$t^r J \subseteq \bigcap_{\mathfrak{P}/\mathfrak{p}} \mathcal{O}_{\mathfrak{P}} = \mathcal{O}'_{\mathfrak{p}}.$$

Hence,

$$t^r \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) = \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(t^r J) \subseteq \mathcal{O}_{\mathfrak{p}} \quad \Longrightarrow \quad v_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J)) \geq -r.$$

In this case, we proved in a previous unit that

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) = t^m \mathcal{O}_{\mathfrak{p}}$$

for some $m \in \mathbb{Z}$. In our case $m \leq -1$ as otherwise $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) \subseteq \mathcal{O}_{\mathfrak{p}}$.

Differentials in extensions

Proof.

Recall that (ω) is the largest divisor in $\mathcal{D}(\mathsf{E})$ on which ω vanishes. Thus, ω does not vanish on $\Lambda_{\mathsf{E}}((\omega)+\mathfrak{p})$. Namely,

$$\exists \alpha \in \Lambda_{\mathsf{E}}((\omega)+\mathfrak{p}) \quad \text{s.t.} \quad \omega(\alpha) \neq 0.$$

Note that $\alpha \notin \Lambda_{\mathsf{E}}((\omega))$.

Since for all other prime divisors $\mathfrak{q} \neq \mathfrak{p}$ we have

$$v_{\mathfrak{q}}((\omega)) = v_{\mathfrak{q}}((\omega)+\mathfrak{p})$$

we conclude that

$$\begin{aligned} v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) &\geq -v_{\mathfrak{p}}((\omega)+\mathfrak{p}) = -v_{\mathfrak{p}}((\omega)) - 1, \\ v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) &\ngeq -v_{\mathfrak{p}}((\omega)), \end{aligned}$$

and so

$$v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) = -v_{\mathfrak{p}}((\omega)) - 1.$$

Differentials in extensions

Proof.

Define $\gamma, \gamma' \in \mathbb{A}_{\mathsf{E}}$ as follows

$$\gamma_{\mathfrak{q}} = \begin{cases} \alpha_{\mathfrak{p}}, & \mathfrak{q} = \mathfrak{p} \\ 0, & \mathfrak{q} \neq \mathfrak{p}. \end{cases} \qquad \gamma'_{\mathfrak{q}} = \begin{cases} 0, & \mathfrak{q} = \mathfrak{p} \\ \alpha_{\mathfrak{q}}, & \mathfrak{q} \neq \mathfrak{p}. \end{cases}$$

Note that

1. γ, γ' are adeles;
2. $\gamma + \gamma' = \alpha$;
3. $\gamma' \in \Lambda_{\mathsf{E}}((\omega))$; and so $\omega(\gamma') = 0$;
4. $\omega(\gamma) = \omega(\alpha) - \omega(\gamma') = \omega(\alpha) \neq 0$.

Write $x = \gamma_{\mathfrak{p}} = \alpha_{\mathfrak{p}}$. Take $y \in \mathsf{E}$ s.t. $\upsilon_{\mathfrak{p}}(y) = \upsilon_{\mathfrak{p}}((\omega))$. Then,

$$\upsilon_{\mathfrak{p}}(xy) = \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(y) = (-\upsilon_{\mathfrak{p}}((\omega)) - 1) + \upsilon_{\mathfrak{p}}((\omega)) = -1 \geq m.$$

Hence, $xy \in t^m \mathcal{O}_{\mathfrak{p}}$.

Differentials in extensions

Proof.

Recall that $\mathrm{Tr}_{F/E}(J) = t^m\mathcal{O}_{\mathfrak{p}}$ and $xy \in t^m\mathcal{O}_{\mathfrak{p}}$, and so

$$\exists z \in J \quad \text{s.t.} \quad \mathrm{Tr}_{F/E}(z) = xy.$$

Define an adele $\beta \in \mathbb{A}_{F/E}$ by

$$\beta_{\mathfrak{P}} = \begin{cases} zy^{-1}, & \mathfrak{P}/\mathfrak{p}; \\ 0, & \text{otherwise.} \end{cases}$$

As $z \in J$ we have that

$$\forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(z) \geq v_{\mathfrak{P}}(\mathrm{Con}_{F/E}(\omega) - \mathfrak{a}').$$

Thus, using that $v_{\mathfrak{p}}(y) = v_{\mathfrak{p}}((\omega))$

$$\begin{aligned} v_{\mathfrak{P}}(\beta) &= v_{\mathfrak{P}}(z) - v_{\mathfrak{P}}(y) \\ &\geq v_{\mathfrak{P}}(\mathrm{Con}_{F/E}(\omega) - \mathfrak{a}') - v_{\mathfrak{P}}(\mathrm{Con}_{F/E}((\omega))) = -v_{\mathfrak{P}}(\mathfrak{a}'). \end{aligned}$$

Differentials in extensions

Proof.

For $\mathfrak{P}$ not over $\mathfrak{p}$,

$$v_{\mathfrak{P}}(\beta) = v_{\mathfrak{P}}(0) = \infty > -v_{\mathfrak{P}}(\mathfrak{a}'),$$

and so $\beta \in \Lambda_{F/E}(\mathfrak{a}')$. Next, we show that $\mathrm{Tr}_{F/E}(\beta) = \gamma$. Indeed,

$$\mathrm{Tr}_{F/E}(\beta)_{\mathfrak{p}} = \mathrm{Tr}_{F/E}(zy^{-1}) = y^{-1}\mathrm{Tr}_{F/E}(z) = y^{-1}yx = \gamma_{\mathfrak{p}}.$$

For $\mathfrak{q} \neq \mathfrak{p}$,

$$\mathrm{Tr}_{F/E}(\beta)_{\mathfrak{q}} = \mathrm{Tr}_{F/E}(0) = 0 = \gamma_{\mathfrak{q}}.$$

Thus,

$$\omega_1(\beta) = \omega(\mathrm{Tr}_{F/E}(\beta)) = \omega(\gamma) \neq 0.$$

□

Differentials in extensions

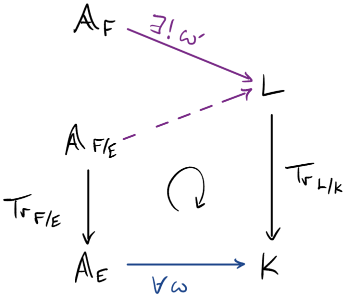
Theorem 7

For every differential ω *of* E/K $\exists!$ *differential* ω' *of* F/L *s.t.*

$$\forall \beta \in \mathbb{A}_{F/E} \qquad \mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$

Furthermore, if $\omega \neq 0$ *then* $\omega' \neq 0$ *and*

$$(\omega') = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$



Differentials in extensions

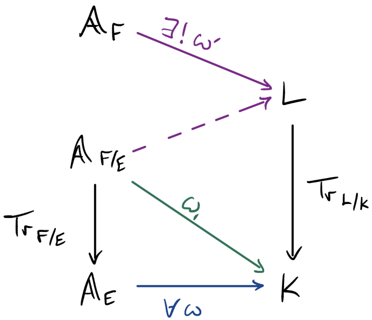
Proof.

Set

$$\mathfrak{a} = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$

Define $\omega_1 : \mathbb{A}_{F/E} \to K$ by

$$\omega_1 = \omega \circ \mathrm{Tr}_{F/E}.$$



Differentials in extensions

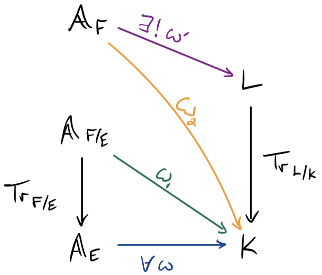
Proof.

Recall Lemma 4 which stated that

$$\forall \mathfrak{b} \in \mathcal{D}(F/L) \qquad \mathbb{A}_F = \mathbb{A}_{F/E} + \Lambda_F(\mathfrak{b}).$$

Using this we will extend ω_1 to $\omega_2 : \mathbb{A}_F \to K$ as follows: Every element of $\mathbb{A}_F$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{F/E}$ and $\gamma \in \Lambda_F(\mathfrak{a})$. We define

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$



Differential

Proof.

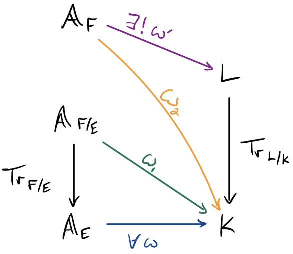
Every element of $\mathbb{A}_F$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{F/E}$ and $\gamma \in \Lambda_F(\mathfrak{a})$. We define

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$

Note that taking $\gamma = 0 \in \Lambda_F(\mathfrak{a})$ we get

$$\omega_2(\beta) = \omega_2(\beta + 0) = \omega_1(\beta),$$

and so ω_2 does indeed extend ω_1.



Differentials in extensions

Proof.

We turn to show that ω_2 is well defined.

If $\beta_1 + \gamma_1 = \beta_2 + \gamma_2$ then

$$\beta_1 - \beta_2 = \gamma_2 - \gamma_1 \in \mathbb{A}_{F/E} \cap \Lambda_F(\mathfrak{a}) = \Lambda_{F/E}(\mathfrak{a}).$$

By Lemma 5, ω_1 is nullified on $\Lambda_{F/E}(\mathfrak{a}) + F$ and so

$$\omega_1(\beta_1) - \omega_1(\beta_2) = \omega_1(\beta_1 - \beta_2) = 0.$$

Therefore,

$$\omega_2(\beta_1 + \gamma_1) = \omega_1(\beta_1) = \omega_1(\beta_2) = \omega_2(\beta_2 + \gamma_2).$$

Hence, ω_2 is well-defined.

Differentials in extensions

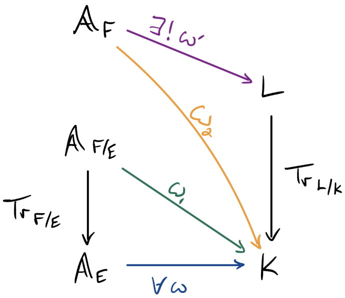
Proof.

Since ω_1 is K-linear so is ω_2. Lemma 1 then implies that

$$\exists! \omega' : \mathbb{A}_F \to L \quad \text{s.t.} \quad \mathrm{Tr}_{L/K} \circ \omega' = \omega_2.$$

We want to show that

$$\forall \beta \in \mathbb{A}_{F/E} \qquad \mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$



Differentials in extensions

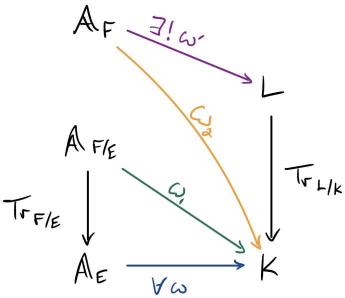
Proof.

We want to show that

$$\forall \beta \in \mathbb{A}_{F/E} \qquad \mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$

For every $\beta \in \mathbb{A}_{F/E}$ we have

$$\mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega_2(\beta) = \omega_1(\beta) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$



Differentials in extensions

Proof.

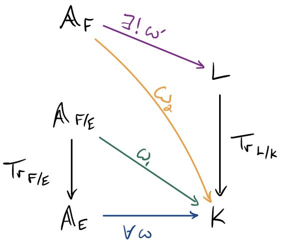
We turn to prove that ω' is a differential. To this end, we will show that ω' vanishes on $\Lambda_F(\mathfrak{a}) + F$.

Otherwise, since $\omega' : \mathbb{A}_F \to L$ is L-linear we will have that

$$\omega'(\Lambda_F(\mathfrak{a}) + F) = L.$$

As $\mathrm{Tr}_{L/K}$ is onto K, we have that

$$\mathrm{Tr}_{L/K}(\omega'(\Lambda_F(\mathfrak{a}) + F)) = K \quad \Longrightarrow \quad \omega_2(\Lambda_F(\mathfrak{a}) + F) = K.$$



Differentials in extensions

Proof.

$$\omega_2(\Lambda_F(\mathfrak{a}) + F) = K. \tag{1}$$

Recall that every element of $\mathbb{A}_F$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{F/E}$ and $\gamma \in \Lambda_F(\mathfrak{a})$, and that we defined

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$

Thus,

$$\omega_2(\Lambda_F(\mathfrak{a})) = \omega_1(0) = 0. \tag{2}$$

Further, by Lemma 5, $\omega_1(F) = 0$. Since $F \hookrightarrow \mathbb{A}_{F/E}$ and ω_2 extend ω_1 on $\mathbb{A}_{F/E}$ we have that

$$\omega_2(F) = 0. \tag{3}$$

Equations (2),(3) imply

$$\omega_2(\Lambda_F(\mathfrak{a}) + F) = 0,$$

in contradiction to Equation (1).

Differentials in extensions

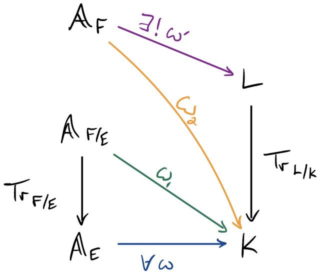
Proof.

We turn to establish uniqueness. Take a differential $\omega'' : \mathbb{A}_F \to L$ s.t.

$$\forall \beta \in \mathbb{A}_{F/E} \qquad \mathrm{Tr}_{L/K}(\omega''(\beta)) = \mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$

Then, $\eta = \omega'' - \omega'$ is a differential of F/L and, in particular is L-linear, so

$$\forall \beta \in \mathbb{A}_{F/E} \qquad \mathrm{Tr}_{L/K}(\eta(\beta)) = \mathrm{Tr}_{L/K}(\omega''(\beta)) - \mathrm{Tr}_{L/K}(\omega'(\beta)) = 0.$$



Differentials in extensions

Proof.

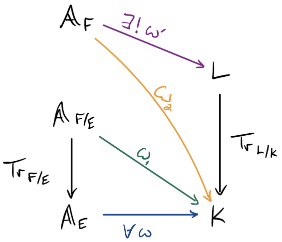
$$\mathrm{Tr}_{L/K}(\eta(\mathbb{A}_{F/E})) = 0.$$

As $\mathrm{Tr}_{L/K}$ is onto, we have that

$$\eta(\mathbb{A}_{F/E}) \subsetneq L.$$

By the L-linearity of η, we get that $\eta(\mathbb{A}_{F/E}) = 0$.

Since η is a differential it also vanishes on some $\Lambda_F(\mathfrak{b})$ for some divisor $\mathfrak{b}$ and so, by Lemma 4, η vanishes on $\mathbb{A}_F$, namely, $\omega' = \omega''$.



Differentials in extensions

Proof.

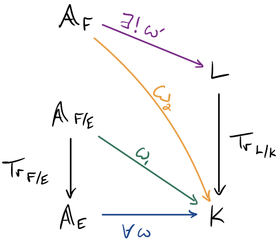
To conclude the proof, we show that

$$(\omega') = \mathfrak{a} = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$

We already proved that ω' vanishes on $\mathfrak{a}$, and so we need to prove that $\mathfrak{a}$ is the largest such divisor.

To this end, take $\mathfrak{a}' \in \mathcal{D}(F)$ s.t. $\mathfrak{a}' \not\leq \mathfrak{a}$. We will show that

$$\exists \beta \in \Lambda_F(\mathfrak{a}') \quad \text{s.t.} \quad \omega'(\beta) \neq 0.$$



Differentials in extensions

Proof.

We wish to show that

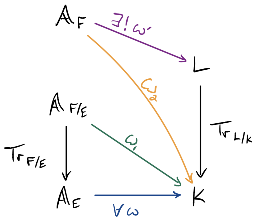
$$\mathfrak{a}' \not\leq \mathfrak{a} \quad \Longrightarrow \quad \exists \beta \in \Lambda_F(\mathfrak{a}') \quad \text{s.t.} \quad \omega'(\beta) \neq 0.$$

By Lemma 6,

$$\exists \beta \in \Lambda_{F/E}(\mathfrak{a}') \subseteq \Lambda_F(\mathfrak{a}') \quad \text{s.t.} \quad \omega_1(\beta) \neq 0.$$

However, $\beta \in \Lambda_{F/E}(\mathfrak{a}')$ and so $\omega_2(\beta) = \omega_1(\beta) \neq 0$.

As $\omega_2(\beta) = \mathrm{Tr}_{L/K}(\omega'(\beta))$ we conclude that $\omega'(\beta) \neq 0$.



Overview

The co-trace

Definition 8

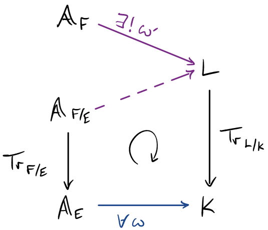
The map

$$\operatorname{cotr}_{F/E} : \Omega_{E/K} \to \Omega_{F/L}$$
$$\omega \mapsto \omega'$$

that is defined implicitly by the property

$$\operatorname{Tr}_{L/K} \circ \operatorname{cotr}_{F/E}(\omega) = \omega \circ \operatorname{Tr}_{F/E}$$

on $\mathbb{A}_{F/E}$ is called the co-trace.



The co-trace

Claim 9

Let $\omega_1, \omega_2 \in \Omega_{E/K}$. Then,

$$\mathrm{cotr}_{F/E}(\omega_1 + \omega_2) = \mathrm{cotr}_{F/E}(\omega_1) + \mathrm{cotr}_{F/E}(\omega_2).$$

Proof.

We have that

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_1) = \omega_1 \circ \mathrm{Tr}_{F/E},$$
$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_2) = \omega_2 \circ \mathrm{Tr}_{F/E}.$$

Thus,

$$\mathrm{Tr}_{L/K} \circ (\mathrm{cotr}_{F/E}(\omega_1) + \mathrm{cotr}_{F/E}(\omega_2)) =$$
$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_1) + \mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_2) =$$
$$\omega_1 \circ \mathrm{Tr}_{F/E} + \omega_2 \circ \mathrm{Tr}_{F/E} = (\omega_1 + \omega_2) \circ \mathrm{Tr}_{F/E},$$

and the proof follows by the (implicit) definition of $\mathrm{cotr}_{F/E}(\omega_1 + \omega_2)$.

The co-trace

Recall that for $\omega \in \Omega_{E/K}$ and $x \in E$, we defined $x\omega \in \Omega_{E/K}$ by

$$\forall \alpha \in \mathbb{A}_E \qquad (x\omega)(\alpha) = \omega(x\alpha).$$

Claim 10

Let $\omega \in \Omega_{E/K}$, and $x \in E$. Then,

$$\mathrm{cotr}_{F/E}(x\omega) = x \cdot \mathrm{cotr}_{F/E}(\omega).$$

Proof.

Let

$$\begin{aligned} \varphi_x : \mathbb{A}_{F/E} &\to \mathbb{A}_{F/E} \\ \alpha &\mapsto x\alpha. \end{aligned}$$

Recall that by the implicit definition of $\mathrm{cotr}_{F/E}$ we have that on $\mathbb{A}_{F/E}$,

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega) = \omega \circ \mathrm{Tr}_{F/E}.$$

The co-trace

Proof.

Thus,

$$\mathrm{Tr}_{\mathsf{L}/\mathsf{K}} \circ \mathrm{cotr}_{\mathsf{F}/\mathsf{E}}(\omega) \circ \varphi_x = \omega \circ \mathrm{Tr}_{\mathsf{F}/\mathsf{E}} \circ \varphi_x.$$

Now, for every $\alpha \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$,

$$(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}} \circ \varphi_x)(\alpha) = \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x\alpha) = x\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha) = (\varphi_x \circ \mathrm{Tr}_{\mathsf{F}/\mathsf{E}})(\alpha).$$

Therefore, on $\mathbb{A}_{\mathsf{F}/\mathsf{E}}$,

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}} \circ \varphi_x = \varphi_x \circ \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}.$$

Thus, on $\mathbb{A}_{\mathsf{F}/\mathsf{E}}$,

$$\mathrm{Tr}_{\mathsf{L}/\mathsf{K}} \circ \mathrm{cotr}_{\mathsf{F}/\mathsf{E}}(\omega) \circ \varphi_x = \omega \circ \varphi_x \circ \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}.$$

The co-trace

Proof.

$$\mathrm{Tr}_{\mathsf{L/K}} \circ \mathrm{cotr}_{\mathsf{F/E}}(\omega) \circ \varphi_x = \omega \circ \varphi_x \circ \mathrm{Tr}_{\mathsf{F/E}}.$$

But

$$x\omega = \omega \circ \varphi_x,$$
$$x \cdot \mathrm{cotr}_{\mathsf{F/E}}(\omega) = \mathrm{cotr}_{\mathsf{F/E}}(\omega) \circ \varphi_x,$$

and so on $\mathbb{A}_{\mathsf{F/E}}$,

$$\mathrm{Tr}_{\mathsf{L/K}} \circ (x \cdot \mathrm{cotr}_{\mathsf{F/E}}(\omega)) = (x\omega) \circ \mathrm{Tr}_{\mathsf{F/E}}.$$

The proof follows by the (implicit) definition of $\mathrm{cotr}_{\mathsf{F/E}}$. □

The co-trace

Claim 11

Let F/E and F'/F be finite separable extensions of function fields. Then,

$$\mathrm{cotr}_{F'/E} = \mathrm{cotr}_{F'/F} \circ \mathrm{cotr}_{F/E}.$$

As with all tower type statement, we omit the proof.

Overview

Hurwitz Genus Formula

Theorem 12

Let F/L *be a finite separable extension of* E/K*. Let* g_E, g_F *be the corresponding genera. Then,*

$$2g_F - 2 = \frac{[F : E]}{[L : K]} \cdot (2g_E - 2) + \deg \mathrm{Diff}(F/E).$$

Hurwitz Genus Formula

$$2g_F - 2 = \frac{[F:E]}{[L:K]} \cdot (2g_E - 2) + \deg \mathrm{Diff}(F/E).$$

Proof.

Take $0 \neq \omega \in \Omega_{E/K}$. By Theorem 7,

$$(\mathrm{cotr}_{F/E}(\omega)) = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$

As $(\omega), (\mathrm{cotr}_{F/E}(\omega))$ are canonical divisors of E/K and F/L, respectively, Riemann-Roch theorem implies that

$$\deg_E((\omega)) = 2g_E - 2 \qquad \deg_F((\mathrm{cotr}_{F/E}(\omega))) = 2g_F - 2.$$

The proof then follows as

$$\deg_F(\mathrm{Con}_{F/E}((\omega))) = \frac{[F:E]}{[L:K]} \cdot \deg_E((\omega)).$$

□