

Algebraic Geometric Codes

Recitation 07b

Shir Peleg

Tel Aviv University

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We have the following diagram

$$\begin{array}{c} E \\ | \\ \mathbb{F}_q(x) \\ | \\ \mathbb{F}_q \end{array}$$

Where the first extension has $tr - deg$ of 1, and the second extension is algebraic.

F is a field, or, $y^p + y - x^{p+1}$ is irreducible

Proof.

From Gauss lemma it is enough to show that the polynomial is irreducible over $\mathbb{F}_q[x][y] \cong \mathbb{F}_q[x, y] \cong \mathbb{F}_q[y][x]$ and thus it is enough to show that the polynomial is irreducible in $\mathbb{F}_q[y][x]$. This follows from Eisenstein's criterion with $p = y$. □

Degree one places

We want to find all the degree one places in F . Note that P has degree one only if $P|_{\mathbb{F}_q(x)}$ has degree one (this is necessary but not sufficient). Recall that the degree one places in $F_q(x)$ correspond to the valuations

$$v_\infty \cup \{v_{x-\alpha} \mid \alpha \in \mathbb{F}_q\}.$$

We need to consider extensions of these valuations.

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Therefore $p \mid v(x)$ denote $v(x) = -\alpha p$. It follows that $v(y) = -\alpha(p+1)$. Up to equivalence (why?) we can assume that $\alpha = 1$.

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$$\deg(P) \cdot [\Gamma(v) : \Gamma(v_\infty)] \leq [F : \mathbb{F}_q(x)]$$

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$$\deg(P) \cdot p \leq p \Rightarrow \deg(P) = 1.$$

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i.e., there are p possible values for $\varphi(y)$ in \mathbb{F}_q such that $\text{Tr}(\varphi(y)) = \alpha'$. Each of these values corresponds to an extension of φ_α , where $L = \mathbb{F}_q$.