Simple Sufficient Conditions for Dedekind-ness

A long time ago we proved:

Theorem

Let $f \in k[x,y]$ irr. Let $C_f = k[x,y]/\langle f \rangle$, $k(Z_f)$ its field of fractions. Then,

$$Z_f(k) \text{ is non-singular} \iff C_f \text{ integrally closed}$$

easy to check/ insure geometric claim

what turned out to be a very useful algebraic property

Now, by Hilber basis theorem, $k[x,y]$ is noetherian $\Rightarrow$ C_f is noetherian.

To conclude that C_f is D.D we need that $\dim C_f = 1$. Recall:

Lemma

A domain. $K = \mathrm{Frac} A$. L/K finite field extension. Let B be the integral closure of A in L. Then, $\dim A = 1 \Rightarrow \dim B = 1$.

C_f is already i.c. If we'll add the assumption that $f(x,y)$ is monic in y then C_f would be integral over $k[x]$ and so the integral closure of $k[x]$ in $k(Z_f)$. By the lemma we conclude:

Proposition

$$\left.\begin{array}{c} f \text{ monic } \& \text{ irreducible} \\ + \\ Z_f(k) \text{ non-singular} \end{array}\right\} \Rightarrow C_f \text{ D.D.}$$

We would like to obtain sufficient conditions for Dedekind-ness in more general settings. In particular, when A is not necessarily $k[x]$ but rather a D.D. In this case $C_f = A[y]/\langle f(y)\rangle$ and we do not have $\frac{\partial f}{\partial x}$ available to us. In this unit we'll obtain sufficient conditions based only on $\frac{\partial f}{\partial y}$ which we'll simply denote by $f'(y)$. To start with lets see what we can say when $A = k[x]$ when we're only allowed to use $\frac{\partial f}{\partial y}$.

Recall that to factor $(x-a)C_f$ we wrote $f(x,y) = (x-a)g(x,y) + \prod_{i=1}^{s}(y-b_i)^{e_i}$ and calculated

$$\frac{\partial f}{\partial y} = (x-a)\frac{\partial g}{\partial y} + \sum_{t=1}^{s} e_t (y-b_t)^{e_t-1} \prod_{j\neq t}(y-b_j)^{e_j}.$$

Thus,

$$\frac{\partial f}{\partial y}(a,b_i) = \begin{cases} \prod_{j\neq i}(b_i-b_j)^{e_j} & e_i = 1 \\ 0 & e_i > 1 \end{cases}$$

In particular

$$\frac{\partial f}{\partial y}(a,b_i) \neq 0 \iff e_i = 1.$$

So, only $\frac{\partial f}{\partial y}$ is used to determine whether $e_i = 1$.

Clearly also

$$e_i = 1 \iff \frac{\partial f}{\partial y}(x,y) \notin M_i = \langle x-a, y-b_i \rangle$$

so $\frac{\partial f}{\partial y}$ does tell us a lot.

The setting

A D.D. $f(y) \in A[y]$ monic, irr. $k = \operatorname{Frac} A$ and $\bar{k}$ algebraic closure of k.

Let $\alpha \in \bar{k}$ be a root of f. Let $M \in \operatorname{Max} A[\alpha]$ and $P = A \cap M$.

($A[\alpha]$ isomorphic to $C_f = A[y]/\langle f(y) \rangle$ where $\alpha \mapsto y + \langle f(y) \rangle$)

Let

$$\bar{f}(y) = \prod_{i=1}^{s} \bar{g}_i(y)^{e_i}$$

be a factorization of $\bar{f}(y) \in (A/P)[y]$ - the reduction of $f(y)$ in $(A/P)[y]$ with $\bar{g}_i(y)$ distinct monic irreducibles. Write

$$f(y) = h(y) + \prod_{i=1}^{s} g_i(y)^{e_i}$$

with $h(y) \in PA[y]$ and $g_i(y)$ s.t. $\bar{g}_i(y)$ is the corresponding reduction.

Let $i_0 \in [s]$ be the unique integer s.t. $M = \langle P, g_{i_0}(\alpha) \rangle$.

(we proved such i_0 exists. It is unique since $\langle P, g_i(\alpha) \rangle$ and $\langle P, g_j(\alpha) \rangle$ are coprime.)

Claim

Under the setting above,

we'll soon show that this implies $e_{M/P}=1$

$$f'(\alpha)\notin M \iff \begin{cases} e_{i_0}=1 & \& \\ A[\alpha]/M \text{ is separable over } A/P. \end{cases}$$

This equivalence motivates our definition of ramification

$$\begin{array}{ccc} \langle P, g_{i_0}(\alpha)\rangle = M & & A[\alpha] \\ \downarrow & & | \\ P & & A \end{array}$$

Proof

In the ring $A[\alpha]$

$$f'(\alpha) = h'(\alpha) + \sum_{i=1}^{s} e_i g_i(\alpha)^{e_i-1} g_i'(\alpha) \prod_{j\neq i} g_j(\alpha)^{e_j}.$$

$h(y)\in PA[y] \implies h'(y)\in PA[y] \implies h'(\alpha)\in PA[\alpha]$ and so

$$f'(\alpha)\notin M \iff \begin{cases} e_{i_0}=1 & \& \\ g_{i_0}'(\alpha)\notin M \end{cases}$$

and so it suffices to show that $g_{i_0}'(\alpha)\notin M \iff A[\alpha]/M$ is separable over A/P.

Let $\bar{\alpha}$ be the image of α in $A[\alpha]/\mathfrak{M}$. Then,

$$A[\alpha]/\mathfrak{M} \cong \left(A[y]/\langle f(y)\rangle\right)/\langle P, g_{i_0}(y)\rangle$$

$$g_{i_0}(y) \mid f(y) \longrightarrow \cong \left(A[y]/\langle g_{i_0}(y)\rangle\right)/P$$

$$\cong (A/P)[y]/\langle \bar{g}_{i_0}(y)\rangle$$

$\bar{g}_{i_0}$ is the min poly of $\bar{\alpha}$ over A/P

$$\cong (A/P)(\bar{\alpha}).$$

$A[\alpha]/\mathfrak{M}$ is separable over A/P $\iff$ $\bar{\alpha}$ is separable over A/P.

$\iff$ $\bar{\alpha}$'s min-poly over A/P - $\bar{g}_{i_0}(y)$ is separable

$\iff$ $\bar{g}_{i_0}'(\bar{\alpha}) \neq 0$ in $A[\alpha]/\mathfrak{M}$.

$\iff$ $g_{i_0}'(\alpha) \notin \mathfrak{M}$.

■

Observe that A D.D $\Rightarrow$ A_P (local) PID. Indeed, we proved that $\dim A = 1 \Rightarrow$ $\dim A_P = 1$ and A noetherian $\Rightarrow$ A_P noetherian. Further A i.c. $\iff$ A_P i.c. $\forall P \in \operatorname{Spec} A$. So, A_P D.D. Recall that we proved that a noetherian local domain of dim $\neq 1$ (such as A_P) is i.c $\iff$ it is a PID.

Claim

With the setting as before, let π be a generator of the max ideal PA_P.

(As explained above, such π exists)

Then,

$$A[\alpha]_M \text{ is a PID} \iff MA[\alpha]_M \text{ can be generated by either } \pi \text{ or by } g_{i_0}(\alpha)$$

To prove the proposition we prove the following lemma.

Lemma

A domain, $\dim A = 1$. Let $M \in \operatorname{Max} A$ s.t. $M = \langle x, y \rangle$. FAE:

1) A_M PID.
2) $\exists u, v \in A$ not both in M s.t. $ux + vy = 0$.
3) Either x or y generates MA_M.

Proof of lemma

1→2: Let π be a generator of MA_M. As $\frac{x}{1}, \frac{y}{1} \in MA_M$, $\exists \frac{a}{r}, \frac{b}{s} \in A_M$ (that is, $a, b \in A$, $r, s \in A \setminus M$) s.t.

$$\frac{x}{1} = \pi \frac{a}{r} \quad \text{and} \quad \frac{y}{1} = \pi \frac{b}{s}$$

and so

$$\Rightarrow \quad brx = \pi ab \quad \text{and} \quad asy = \pi ab$$

$$\Rightarrow \quad (br)x + (-as)y = 0.$$

Now, if both $a, b \in M$ then $\frac{x}{1} = \frac{a}{r}\pi \in (MA_M)^2$ and $\frac{y}{1} = \frac{b}{s}\pi \in (MA_M)^2$. Thus, $\langle \frac{x}{1}, \frac{y}{1} \rangle \subseteq \langle \pi \rangle^2$. However, $M = \langle x, y \rangle \Rightarrow MA_M = \langle \frac{x}{1}, \frac{y}{1} \rangle$ which implies $\langle \pi \rangle = \langle \pi \rangle^2$. As π not a unit, this stands in contradiction to A_M being a PID.

Now, since $r, s \notin M$ and M prime, $a \in M \iff -as \in M$ and $b \in M \iff br \in M$ and so not both of br and $-as$ can be in M.

2 → 3:

If $ux + vy = 0$ in A with (wlog) $u \notin M$ then in A_M, $\frac{x}{1} = -\frac{v}{u} \cdot \frac{y}{1}$ and so $MA_M = \langle \frac{x}{1}, \frac{y}{1} \rangle = \langle \frac{y}{1} \rangle$.

3 → 1: Since A_M is a local ring with max ideal MA_M, and since $\dim A = 1$, MA_M is the only nonzero prime ideal of A_M. We assume it is principal and so by a neat lemma we proved, A_M is a PID. ∎

Proof of Claim

As $M = \langle P, g_{i_0}(\alpha) \rangle$, $MA[\alpha]_M = \langle \pi, g_{i_0}(\alpha) \rangle$. Now, $\dim A[\alpha]_M = 1$ since $\dim A = 1$ and $A[\alpha]$ is an algebraic extension so $\dim A[\alpha] = 1$. Thus we can apply the lemma to conclude that

$$A[\alpha]_M \text{ PID} \iff \text{either } \pi \text{ or } g_{i_0}(\alpha) \text{ generates } MA[\alpha]_M.$$ ∎

Claim

If $e_{i_0} = 1$ then $A[\alpha]_M$ is a PID and $MA[\alpha]_M = \langle \pi \rangle$. In particular, $e_{M|P} = 1$.

Proof

Since $MA[\alpha]_M$ is the only nonzero prime ideal of $A[\alpha]_M$ ($A[\alpha]_M$ being a local ring of dim 1), showing that $MA[\alpha]_M = \langle \pi \rangle \Rightarrow A[\alpha]_M$ PID.

Note that $h(y) \in PA[y] \Rightarrow h(y) = \sum_i p_i h_i(y)$ $\quad h_i(y) \in A[y]$ $\quad p_i \in P$

in A_P $\Rightarrow$ $h(y) = \sum_i p_i\, h_i(y)$

$p_i \in PA_P = \pi A_P$ $\Rightarrow$ $h(y) = \sum_i \pi \cdot t_i(y)$ $\quad t_i(y) \in A_P[y]$

$\Rightarrow$ $h(\alpha) = \sum_i \pi\, t_i(\alpha)$

$\Rightarrow$ $h(\alpha) \in \pi\, A[\alpha]_M$

Now, in $A[\alpha]$:

$$0 = f(\alpha) = h(\alpha) + g_{i_0}(\alpha)^{e_{i_0}} \cdot \prod_{j \neq i_0} g_j(\alpha)^{e_j}$$

For $j \neq i_0$ $g_j(\alpha) \notin M$ and so $\prod_{j \neq i_0} g_j(\alpha)^{e_j}$ is a unit in $A[\alpha]_M$. Hence,

$$g_{i_0}(\alpha)^{e_{i_0}} \in h(\alpha)\, A[\alpha]_M \subseteq \pi\, A[\alpha]_M.$$

Using our assumption $e_{i_0}=1$ we get that $MA[\alpha]_M = \pi A[\alpha]_M$ as desired.

As $MA[\alpha]_M = \pi A[\alpha]_M = PA[\alpha]_M$ we conclude $e_{M|P}=1$. ∎

Claim

$A[\alpha]$ is a D.D. $\iff$ $A[\alpha]_M$ is a PID $\forall$ max ideals M in $A[\alpha]$ that contain $f'(\alpha)$. Further, when $f(y)$ is separable, there are only finitely many $M \in \operatorname{Max} A[\alpha]$ that contain $f'(\alpha)$.

Proof

Take $M \in \operatorname{Max} A[y]$ with $f'(\alpha) \notin M$. By a previous claim, $e_{i_0}=1$ for the corresponding e_{i_0} and s., by another previous claim, $A[\alpha]_M$ is a PID. We thus need to prove that

$$A[\alpha] \text{ D.D.} \iff A[\alpha]_M \text{ is a PID } \forall M \in \operatorname{Max} A[y].$$

which we've already established (based on the fact that in a noetherian local domain of dim 1, i.e $\iff$ PID).

Now, if $f(y)$ is separable and since $f(\alpha)=0$ we get $f'(\alpha)\neq 0$. $A[\alpha]$ is noetherian and $\dim A[\alpha]=1$ (since $A[\alpha]$ is integral over A and $\dim A=1$). We proved that under these two assumptions, any ideal, and in particular $f'(\alpha)A[\alpha]$, is contained in only finitely many max ideals.

After all this hard work we get

Corollary

A D.D. $f(y)\in A[y]$ monic, irr. Let α denote the class of y in $C_f = A[y]/\langle f(y)\rangle$. (Alternatively, $\alpha\in k$ is a root of f)

Let $M\in \operatorname{Max} C_f$.

1) $f'(\alpha)\notin M \Longrightarrow (C_f)_M$ PID and M is unramified over A.

2) If C_f D.D then M is ramified over $A \iff f'(\alpha)=M$.

Proof

(1) readily follows by the claims we proved.

(2) We proved in a previous unit that if C_f is a D.D. then what we denoted as e_i, (the exponent in the polynomial factorization in the corresponding reduction) is equal to $e_{m/p}$. The proof then follows by a previous claim.

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