AG codes - Spring 2022
Problem Set 03

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Problem 1. *Let F/K be a function field.*

(a) $\Lambda(\mathfrak{a})$ *is a K-vector space, a subspace of* $\mathbb{A}$.

(b) $\mathcal{L}(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap F$.

Problem 2. *For divisors $\mathfrak{a}, \mathfrak{b}$ and $x \in F$. Show*

(a) $\mathfrak{a} \leq \mathfrak{b} \to \Lambda(\mathfrak{a}) \subseteq \Lambda(b)$.

(b) $\Lambda(\mathfrak{a}) \cap \Lambda(\mathfrak{b}) = \Lambda(\min(\mathfrak{a}, \mathfrak{b}))$.

(c) $\Lambda(\mathfrak{a}) + \Lambda(\mathfrak{b}) = \Lambda(\max(\mathfrak{a}, \mathfrak{b}))$.

(d) $x\Lambda(\mathfrak{a}) = \Lambda(\mathfrak{a} - (x))$.

Problem 3. *Consider the function field $K(x)/K$*

(a) *Let $f = \frac{g}{h} \in K(x)$, use unique factorization to describe (f).*

(b) *Prove that the genus of $K(x)/K$ is 0, and that every divisor of degree -2 is canonical.*

(c) *Prove that every degree 0 divisor is principle.*

(d) *Let $\mathfrak{a} \in \mathcal{P}$, prove that* $\dim \mathfrak{a} = \max(0, \deg \mathfrak{a} + 1)$.

(e) *Let $f \in K[x]$ be a polynomial. Show that* $\deg(f(x))_\infty = \deg f$.

Problem 4. *Let F/K be a function field.*

(a) *Prove that if $x, y \in F \setminus K$ are such that $\deg(x)_\infty, \deg(y)_\infty$ are co-prime then $F = K(x, y)$.*

(b) *Assume F/K has a degree one prime divisor. Prove that there exist $x, y \in F$ such that $[F : K(x)] = [F : K(y)] = 2g + 1$ and $F = K(x, y)$.*

Problem 5. *Assume that char$(K) \neq 2$. Let $F = K(x, y)$ with*

$$y^2 = f(x) \in K[x]\ ,\ \deg f(x) = 2m + 1 \geq 3.$$

Show:

(a) *If $g(t)$ is irreducible in $K[t]$, then it is also irreducible in $k(x)[t]$.*

(b) *K is the full constant field of F.*

(c) *There is exactly one place $P \in \mathcal{P}$ which is a pole of x, and this place is also the only pole of y.*

(d) *For every $r \geq 0$, the elements $1, x, x^2, \ldots, x^r, y, xy, ..., x^s y$ with $0 \leq s < r - m$ are in $\mathcal{L}(2rP)$.*

(e) *The genus of F/K satisfies $g \leq m$.*