Free Probability and Ramanujan Graphs - HW #3

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Exercises with * will be graded. Submission in singles or pairs. Contact Gal for questions and clarifications.

1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $(\mathcal{A}_i)_{i\in I}$ be freely independent unital subalgebras of $\mathcal{A}$. Let $a_1, \ldots, a_k$ be centered elements of $\mathcal{A}_{i_1}, \ldots, \mathcal{A}_{i_k}$ such that $i_1 \neq i_2 \neq \ldots \neq i_{k-1} \neq i_k$. Similarly, let $b_1, \ldots, b_l$ be centered elements of $\mathcal{A}_{j_1}, \ldots, \mathcal{A}_{j_l}$ such that $j_1 \neq j_2 \neq \ldots \neq j_{l-1} \neq j_l$.

   (a) Prove that

   $$\varphi(a_1 \cdots a_k b_l \cdots b_1) = \begin{cases} \varphi(a_1 b_1) \cdots \varphi(a_k b_k), & k = l \text{ and } i_1 = j_1, \ldots, i_k = j_k \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $\bigcup_{i\in I} \mathcal{A}_i$. Prove that if $\varphi_{\mathcal{A}_i}$ is a trace for every $i \in I$ then $\varphi_{\mathcal{B}}$ is a trace.

2. * Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a, u \in \mathcal{A}$ be free elements such that u is a Haar unitary and $\varphi(a) = \alpha \in \mathbb{C}$. Prove that for every $n \geq 0$, $\varphi\left(u^{-n}(ua)^n\right) = \alpha^n$.

3. Let G be a group with identity element denoted e, and $(G_i)_{i\in I}$ be subgroups of G. The subgroups are said to be *free* if

   $$g_1 g_2 \cdots g_k \neq e$$

   whenever $g_i \neq e$ for all $i \in [k]$, and no consecutive g_i, g_{i+1} come from the same subgroup (informally, there are no non-trivial relations between the groups).

   Prove that the following are equivalent:

   (a) The subgroups $(G_i)_{i\in I}$ are free.

   (b) The algebras $(\mathbb{C}G_i)_{i\in I}$ (considered as subalgebras of $\mathbb{C}G$) are freely independent in the non-commutative probability space $(\mathbb{C}G, \tau_G)$.

4. * Let $(\mathcal{A}, \varphi)$ and $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N})$ be non commutative probability spaces. Consider $a, b \in \mathcal{A}$ and $a_N, b_N \in \mathcal{A}_N$ such that $(a_N, b_N) \xrightarrow{\text{distr}} (a, b)$, and that for every N, the random variables (a_N, b_N) are free (with respect to φ_N). Prove that a and b are free with respect to φ.