# Hurwitz Genus Formula

Unit 22

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# Overview

# Hurwitz Genus Formula

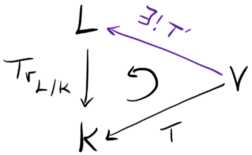
Throughout this unit F/L is a separable finite extension of E/K.

**Lemma 1**

*Let* L/K *be a finite separable field extension. Let* $V$ *be an* L*-vector space (and so* $V$ *is also a* K*-vector space). Let* $T : V \to \mathrm{K}$ *be a* K*-linear map.*

*Then,* $\exists!\, T' : V \to \mathrm{L}$ *that is* L*-linear s.t.*

$$T' \circ \mathrm{Tr}_{\mathrm{L}/\mathrm{K}} = T.$$




We omit the proof of this fact.

# Adeles - recall

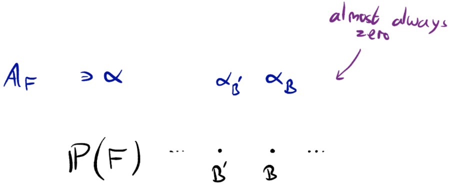
Recall that an adele of F/L is a function $\alpha : \mathbb{P}(F/L) \to F$ that maps $\mathfrak{P} \to \alpha_{\mathfrak{P}}$ s.t. $v_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \geq 0$ almost always.

The set of adeles of F/L is denoted by $\mathbb{A}_{F/L}$ or $\mathbb{A}_F$. Recall that $\mathbb{A}_F$ is an F-algebra. Multiplying by elements of F is done via the embedding $F \hookrightarrow \mathbb{A}_F$ where $x \mapsto [x]$ in which $[x]_{\mathfrak{P}} = x$.

For $D \in \mathcal{D}(F/L)$ we defined

$$\Lambda_F(D) = \{\alpha \in \mathbb{A}_F \mid \forall \mathfrak{P} \in \mathbb{P}(F/L) \quad v_{\mathfrak{P}}(\alpha) + v_{\mathfrak{P}}(D) \geq 0\}.$$

We sometimes write $\Lambda(D)$ for short.



# Adeles of extensions

## Definition 2

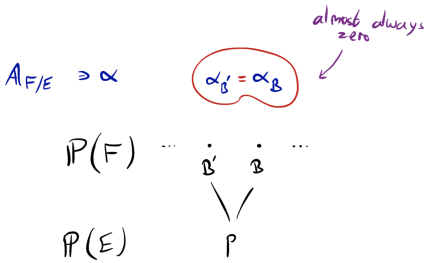
We extend the above definition to extensions.

$$\mathbb{A}_{\mathsf{F}/\mathsf{E}} = \{\alpha \in \mathbb{A}_{\mathsf{F}} \mid \mathfrak{P}_1 \cap \mathsf{E} = \mathfrak{P}_2 \cap E \implies \alpha_{\mathfrak{P}_1} = \alpha_{\mathfrak{P}_2}\}.$$

Note that $\mathsf{F} \hookrightarrow \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ and so $\mathbb{A}_{\mathsf{F}/\mathsf{E}}$ is an F-subalgebra of $\mathbb{A}_{\mathsf{F}}$.

Moreover, for $D \in \mathcal{D}(\mathsf{F}/\mathsf{L})$ we define

$$\Lambda_{\mathsf{F}/\mathsf{E}}(D) = \mathbb{A}_{\mathsf{F}/\mathsf{E}} \cap \Lambda(D).$$



# Adeles of extensions

**Definition 3**

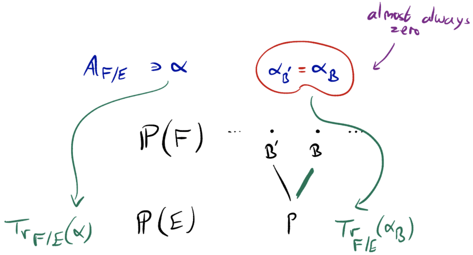
We extend $\mathrm{Tr}_{F/E} : F \to E$ to the map

$$\mathrm{Tr}_{F/E} : \mathbb{A}_{F/E} \to \mathbb{A}_E$$

as follows: For $\alpha \in \mathbb{A}_{F/E}$ and $\mathfrak{p} \in \mathbb{P}(E)$,

$$(\mathrm{Tr}_{F/E}(\alpha))_{\mathfrak{p}} = \mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}})$$

where $\mathfrak{P}$ is some prime divisor lying over $\mathfrak{p}$.



We need to prove that indeed $\mathrm{Tr}_{F/E}(\alpha) \in \mathbb{A}(E)$. Namely, we need to show that $\mathrm{Tr}_{F/E}(\alpha)_{\mathfrak{p}} \geq 0$ almost always.

As $\alpha \in \mathbb{A}_{F/E}$ we have that $\alpha_{\mathfrak{P}} \geq 0$, or equivalently $\alpha_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}$, almost always. Thus, for almost all $\mathfrak{p} \in \mathbb{P}(E)$ every $\mathfrak{P}/\mathfrak{p}$ is s.t. $\alpha_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}$. For every such $\mathfrak{p}$,

$$\alpha_{\mathfrak{P}} \in \bigcap_{\mathfrak{P}/\mathfrak{p}} \mathcal{O}_{\mathfrak{P}} = \mathcal{O}'_{\mathfrak{p}}.$$

Recall that $\mathrm{Tr}_{F/E}(\mathcal{O}'_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{p}}$, and so for almost all $\mathfrak{p}$,

$$(\mathrm{Tr}_{F/E}(\alpha))_{\mathfrak{p}} = \mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}}) \in \mathcal{O}_{\mathfrak{p}},$$

thus establishing that $\mathrm{Tr}_{F/E}(\alpha) \in \mathbb{A}(E)$.

# Adeles of extensions

We further remark that

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}([x]) = [\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x)].$$

**Lemma 4**

*For every $\mathcal{D} \in \mathcal{D}(\mathsf{F}/\mathsf{L})$ we have that*

$$\mathbb{A}_{\mathsf{F}} = \mathbb{A}_{\mathsf{F}/\mathsf{E}} + \Lambda(D).$$

**Proof.**

The inclusion $\mathbb{A}_{\mathsf{F}} \supset \mathbb{A}_{\mathsf{F}/\mathsf{E}} + \Lambda(D)$ is obvious. For the other inclusion, take $\alpha \in \mathbb{A}_{\mathsf{F}}$. We first construct some $\beta \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ as follows.

# Adeles of extensions

**Proof.**

Take $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$. The set of $\mathfrak{P}/\mathfrak{p}$ is finite and so by WAT, $\exists x_{\mathfrak{p}} \in \mathsf{F}$ s.t.

$$\forall \mathfrak{P}/\mathfrak{p} \quad \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}} - x_{\mathfrak{p}}) \geq -v_{\mathfrak{P}}(D).$$

Note that for all most all $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$ we have that $\forall \mathfrak{P}/\mathfrak{p} \ \upsilon_{\mathfrak{P}}(D) = 0$. Moreover, since $\alpha \in \mathbb{A}_{\mathsf{F}}$, for almost all $\mathfrak{P} \in \mathbb{P}(\mathsf{F})$, $\upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \geq 0$. Thus, for almost all $\mathfrak{p}$,

$$\begin{aligned} \upsilon_{\mathfrak{P}}(x_{\mathfrak{p}}) &= \upsilon_{\mathfrak{P}}(x_{\mathfrak{p}} - \alpha_{\mathfrak{P}} + \alpha_{\mathfrak{P}}) \\ &\geq \min(\upsilon_{\mathfrak{P}}(x_{\mathfrak{p}} - \alpha_{\mathfrak{P}}), \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}})) \\ &\geq 0. \end{aligned}$$

# Adeles of extensions

**Proof.**

With this, we define $\beta : \mathbb{P}(\mathsf{F}/\mathsf{L}) \to \mathsf{F}$ by

$$\beta_{\mathfrak{P}} = x_{\mathfrak{p}},$$

where $\mathfrak{p}$ is the prime divisor lying under $\mathfrak{P}$.

$\beta \in \mathbb{A}_{\mathsf{F}}$ as $v_{\mathfrak{P}}(\beta_{\mathfrak{P}}) = v_{\mathfrak{P}}(x_{\mathfrak{p}}) \geq 0$ almost always. Moreover, $\beta \in \mathbb{A}(\mathsf{F}/\mathsf{E})$ since $\beta_{\mathfrak{P}} = x_{\mathfrak{p}} = \beta_{\mathfrak{P}'}$ for all places $\mathfrak{P}, \mathfrak{P}'$ lying over $\mathfrak{p}$.

Lastly, note that $\alpha - \beta \in \Lambda(D)$. Indeed, $\forall \mathfrak{P} \in \mathbb{P}(\mathsf{F})$,

$$v_{\mathfrak{P}}(\alpha - \beta) = v_{\mathfrak{P}}(\alpha_{\mathfrak{P}} - \beta_{\mathfrak{P}}) = v_{\mathfrak{P}}(\alpha_{\mathfrak{P}} - x_{\mathfrak{p}}) \geq -v_{\mathfrak{P}}(D).$$

Thus,

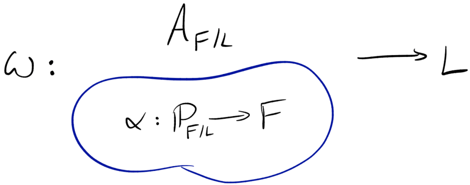
$$\alpha = \beta + (\alpha - \beta) \in \mathbb{A}_{\mathsf{F}/\mathsf{E}} + \Lambda(D),$$

concluding the proof. □

# Overview

# Differential - recall

Recall that a differential of F/L is an L-linear map $\omega : \mathbb{A}_F \to L$ that is nullified on a subspace of the form $\Lambda(\mathcal{D}) + F$ for some divisor $D$.




For a differential $\omega \neq 0$ we defined the canonical divisor

$$(\omega) = \max\{D \in \mathcal{D}(F/L) \mid \omega|_{\Lambda(D)+F} = 0\}.$$

In particular, $\omega|_{\Lambda((\omega))} = 0$.

# Differentials in extensions

**Lemma 5**

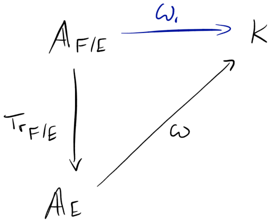
*Let* $\omega : \mathbb{A}_E \to K$ *be a differential of* $E/K$. *We define a map*

$$\omega_1 : \mathbb{A}_{F/E} \to K$$

*by* $\omega_1 = \omega \circ \mathrm{Tr}_{F/E}$. *Then,*

1. $\omega_1$ *is* K*-linear; and*
2. $\omega_1$ *is nullified on* $\Lambda_{F/E}(D) + F$, *where*

$$D = \mathrm{Con}_{F/E}(\omega) + \mathrm{Diff}(F/E).$$



# Differentials in extensions

**Proof.**

The first item follows since both $\mathrm{Tr}_{F/E}$ and $\omega$ are K-linear maps.

For the second item, first note that $\omega_1|_F = 0$. Indeed, $\mathrm{Tr}_{F/E}(F) = E$, and $\omega|_E = 0$. We turn to prove that $\omega|_{\Lambda_{F/E}(D)} = 0$.

Take $\alpha \in \Lambda_{F/E}(D)$. We need to show that $\omega(\mathrm{Tr}_{F/E}(\alpha)) = 0$. To this end we show that

$$\mathrm{Tr}_{F/E}(\alpha) \in \Lambda((\omega)).$$

Equivalently,

$$\forall \mathfrak{p} \in \mathbb{P}(E/K) \quad v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\alpha)) + v_{\mathfrak{p}}((\omega)) \geq 0.$$

Thus, we need to show that for all $\mathfrak{p}$ and $\mathfrak{P}/\mathfrak{p}$,

$$v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\alpha_{\mathfrak{P}})) + v_{\mathfrak{p}}((\omega)) \geq 0.$$

# Differentials in extensions

**Proof.**

Fix $\mathfrak{p}$ and let $x \in \mathsf{E}$ be s.t. $\upsilon_{\mathfrak{p}}(x) = \upsilon_{\mathfrak{p}}((\omega))$. Then, for all $\mathfrak{P}/\mathfrak{p}$

$$
\begin{aligned}
\upsilon_{\mathfrak{P}}(x\alpha_{\mathfrak{P}}) &= \upsilon_{\mathfrak{P}}(x) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&= e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&= e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{P}}(\alpha_{\mathfrak{P}}) \\
&\geq e(\mathfrak{P}/\mathfrak{p})\upsilon_{\mathfrak{p}}((\omega)) - \upsilon_{\mathfrak{P}}(D) \\
&= \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D) \\
&= \upsilon_{\mathfrak{P}}(-\mathrm{Diff}(\mathsf{F}/\mathsf{E})) \\
&= -d(\mathfrak{P}/\mathfrak{p}).
\end{aligned}
$$

Thus, $x\alpha_{\mathfrak{P}} \in \mathsf{C}_{\mathfrak{p}}$.

# Differentials in extensions

**Proof.**

Fix $\mathfrak{p}$ and let $x \in \mathsf{E}$ be s.t. $\upsilon_{\mathfrak{p}}(x) = \upsilon_{\mathfrak{p}}((\omega))$. Then, $x\alpha_{\mathfrak{P}} \in \mathsf{C}_{\mathfrak{p}}$. Thus,

$$\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x\alpha_{\mathfrak{P}})) \geq 0.$$

Since $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}$ is E-linear, we get that

$$\begin{aligned} \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(x\alpha_{\mathfrak{P}})) &= \upsilon_{\mathfrak{p}}(x\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \\ &= \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \\ &= \upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})). \end{aligned}$$

Thus,

$$\upsilon_{\mathfrak{p}}((\omega)) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\alpha_{\mathfrak{P}})) \geq 0$$

which, recall, concludes the proof. □

# Differentials in extensions

Let $\omega : \mathbb{A}_{\mathsf{E}} \to \mathsf{K}$ be a differential of $\mathsf{E}/\mathsf{K}$. Recall that we have defined the map

$$\omega_1 : \mathbb{A}_{\mathsf{F}/\mathsf{E}} \to \mathsf{K}$$

by $\omega_1 = \omega \circ \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}$. We further denoted

$$D = \mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) + \mathrm{Diff}(\mathsf{F}/\mathsf{E}).$$

**Lemma 6**

$\forall D' \in \mathcal{D}(\mathsf{F})$ *such that* $D' \not\leq D$ $\exists \beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(D')$ *such that* $\omega_1(\beta) \neq 0$.

# Differentials in extensions

## Lemma 7

$\forall D' \in \mathcal{D}(\mathsf{F})$ *such that* $D' \not\leq D$ $\exists \beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(D')$ *such that* $\omega_1(\beta) \neq 0$.

## Proof.

Since $D' \not\leq D$ there is $\mathfrak{P}' \in \mathbb{P}(\mathsf{F}/\mathsf{L})$ s.t.

$$v_{\mathfrak{P}'}(D') > v_{\mathfrak{P}'}(D) = v_{\mathfrak{P}'}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega)) + d(\mathfrak{P}'/\mathfrak{p}),$$

where $\mathfrak{p}$ is the place lying under $\mathfrak{P}'$. That is,

$$v_{\mathfrak{P}'}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D') < -d(\mathfrak{P}'/\mathfrak{p}).$$

Define

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(z) \geq v_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D')\}.$$

$J$ is closed under addition and under multiplication by $\mathcal{O}'_{\mathfrak{p}}$ and so $J$ is an $\mathcal{O}'_{\mathfrak{p}}$-module. Furthermore, $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J)$ is an $\mathcal{O}_{\mathfrak{p}}$-module.

# Differentials in extensions

**Proof.**

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \;\; \upsilon_{\mathfrak{P}}(z) \geq \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D')\}.$$

By WAT $\exists z' \in J$ s.t.

$$\forall \mathfrak{P}/\mathfrak{p} \quad \upsilon_{\mathfrak{P}}(z) = \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D')$$

In particular,

$$\upsilon_{\mathfrak{P}'}(z') < -d(\mathfrak{P}'/\mathfrak{p}),$$

and so $z' \notin \mathsf{C}_{\mathfrak{p}}$. Thus, $\exists v \in \mathcal{O}'_{\mathfrak{p}}$ s.t.

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(vz') \notin \mathcal{O}_{\mathfrak{p}}.$$

As $J$ is an $\mathcal{O}'_{\mathfrak{p}}$-module, $vz' \in J$ and so $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) \not\subseteq \mathcal{O}_{\mathfrak{p}}$. In particular, $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) \neq \{0\}$.

# Differentials in extensions

**Proof.**

$$J = \{z \in \mathsf{F} \mid \forall \mathfrak{P}/\mathfrak{p} \;\; \upsilon_{\mathfrak{P}}(z) \geq \upsilon_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D')\}.$$

Let $t \in \mathsf{E}$ be with $\upsilon_{\mathfrak{p}}(t) = 1$. Thus, for a sufficiently large $r$,

$$t^r J \subseteq \bigcap_{\mathfrak{P}/\mathfrak{p}} \mathcal{O}_{\mathfrak{P}} = \mathcal{O}'_{\mathfrak{p}}$$

Thus,

$$t^r \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) = \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(t^r J) \subseteq \mathcal{O}_{\mathfrak{p}} \quad \Longrightarrow \quad \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J)) \geq -r.$$

In this case, we proved that

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) = t^m \mathcal{O}_{\mathfrak{p}}$$

for some $m \in \mathbb{Z}$. In our case $m \leq -1$ as otherwise $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) \subseteq \mathcal{O}_{\mathfrak{p}}$.

# Differentials in extensions

**Proof.**

Recall that $(\omega)$ is the largest divisor in $\mathcal{D}(\mathsf{E}/\mathsf{K})$ on which $\omega$ vanishes. Thus, $\omega$ does not vanish on $\Lambda_{\mathsf{E}}((\omega)+\mathfrak{p})$. Namely,

$$\exists \alpha \in \Lambda_{\mathsf{E}}((\omega)+\mathfrak{p}) \quad \text{s.t.} \quad \omega(\alpha) \neq 0.$$

Note that $\alpha \notin \Lambda_{\mathsf{E}}((\omega))$.

Since for all other prime divisors $\mathfrak{q} \neq \mathfrak{p}$ we have

$$v_{\mathfrak{q}}((\omega)) = v_{\mathfrak{q}}((\omega)+\mathfrak{p})$$

we conclude that

$$v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) \geq -v_{\mathfrak{p}}((\omega)+\mathfrak{p}) = -v_{\mathfrak{p}}((\omega)) - 1,$$
$$v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) \not\geq -v_{\mathfrak{p}}((\omega)),$$

and so

$$v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) = -v_{\mathfrak{p}}((\omega)) - 1.$$

# Differentials in extensions

**Proof.**

Define $\gamma, \gamma' \in \mathbb{A}_E$ from $\alpha$ by setting

$$\gamma_{\mathfrak{q}} = \begin{cases} \alpha_{\mathfrak{p}}, & \mathfrak{q} = \mathfrak{p} \\ 0, & \text{otherwise} \end{cases} \qquad \gamma'_{\mathfrak{q}} = \begin{cases} 0, & \mathfrak{q} = \mathfrak{p} \\ \alpha_{\mathfrak{q}}, & \text{otherwise.} \end{cases}$$

Note that

1. $\gamma, \gamma'$ are adeles;
2. $\gamma + \gamma' = \alpha$;
3. $\gamma' \in \Lambda_E((\omega))$; and so $\omega(\gamma') = 0$.
4. $\omega(\gamma) = \omega(\alpha) - \omega(\gamma') = \omega(\alpha) \neq 0$.

Write $x = \gamma_{\mathfrak{p}} = \alpha_{\mathfrak{p}}$. Take $y \in E$ s.t. $v_{\mathfrak{p}}(y) = v_{\mathfrak{p}}((\omega))$. Then,

$$v_{\mathfrak{p}}(xy) = v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(y) = (-v_{\mathfrak{p}}((\omega)) - 1) + v_{\mathfrak{p}}((\omega)) = -1 \geq m.$$

Hence, $xy \in t^m \mathcal{O}_{\mathfrak{p}}$.

# Differentials in extensions

**Proof.**

Recall that $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(J) = t^m \mathcal{O}_{\mathfrak{p}}$ and $xy \in t^m \mathcal{O}_{\mathfrak{p}}$, and so $\exists z \in J$ s.t. $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(z) = xy$. Define an adele $\beta \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ by

$$\beta_{\mathfrak{P}} = \begin{cases} zy^{-1}, & \mathfrak{P}/\mathfrak{p} \\ 0, & \text{otherwise.} \end{cases}$$

As $z \in J$ we have that

$$\forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(z) \geq v_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D').$$

Thus,

$$\begin{aligned} v_{\mathfrak{P}}(\beta) &= v_{\mathfrak{P}}(z) - v_{\mathfrak{P}}(y) \\ &\geq v_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}(\omega) - D') - v_{\mathfrak{P}}(\mathrm{Con}_{\mathsf{F}/\mathsf{E}}((\omega))) \\ &= -v_{\mathfrak{P}}(D') \end{aligned}$$

# Differentials in extensions

**Proof.**

For $\mathfrak{P}$ not over $\mathfrak{p}$,

$$v_{\mathfrak{P}}(\beta) = v_{\mathfrak{P}}(0) = \infty > -v_{\mathfrak{P}}(D'),$$

and so $\beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(D')$. Next, we show that $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\beta) = \gamma$. Indeed,

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\beta)_{\mathfrak{p}} = \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zy^{-1}) = y^{-1}\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(z) = y^{-1}yx = \gamma_p.$$

For $\mathfrak{q} \neq \mathfrak{p}$,

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\beta)_{\mathfrak{q}} = \mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(0) = 0 = \gamma_{\mathfrak{q}}.$$

Thus,

$$\omega_1(\beta) = \omega(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(\beta)) = \omega(\gamma) \neq 0.$$

□

# Differentials in extensions

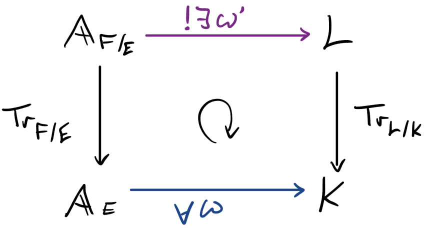
**Theorem 8**

*For every differential* $\omega$ *of* $E/K$ $\exists!$ *differential* $\omega'$ *of* $F/L$ *s.t.* $\forall \beta \in \mathbb{A}_{F/E}$,

$$\mathrm{Tr}_{L/K}(\omega'(\beta)) = \omega(\mathrm{Tr}_{F/E}(\beta)).$$

*Furthermore, if* $\omega \neq 0$ *then* $\omega' \neq 0$ *and*

$$(\omega') = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$



# Differentials in extensions

**Proof.**

Set

$$D = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$

Define $\omega_1 : \mathbb{A}_{F/E} \to K$ by

$$\omega_1 = \omega \circ \mathrm{Tr}_{F/E}.$$

Recall Lemma 4 which stated that

$$\forall \mathcal{D} \in \mathcal{D}(F/L) \qquad \mathbb{A}_F = \mathbb{A}_{F/E} + \Lambda_F(D).$$

Using this we will extend $\omega_1$ to $\omega_2 : \mathbb{A}_F \to K$ as follows.

Every element of $\mathbb{A}_F$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{F/E}$ and $\gamma \in \Lambda_F(D)$. We define

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$

# Differential

**Proof.**

Every element of $\mathbb{A}_\mathsf{F}$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ and $\gamma \in \Lambda_\mathsf{F}(D)$. We define

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$

Note that taking $\gamma = 0 \in \Lambda_\mathsf{F}(D)$ we get

$$\omega_2(\beta + 0) = \omega_2(\beta) = \omega_1(\beta)$$

and so $\omega_2$ does indeed extend $\omega_1$.

# Differentials in extensions

**Proof.**

We turn to show that $\omega_2$ is well defined.

If $\beta_1 + \gamma_1 = \beta_2 + \gamma_2$ then

$$\beta_1 - \beta_2 = \gamma_2 - \gamma_1 \in \mathbb{A}_{F/E} \cap \Lambda_F(D) = \Lambda_{F/E}(D).$$

By Lemma 5,

$$\omega_1(\beta_1) - \omega_1(\beta_2) = \omega_1(\beta_1 - \beta_2) = 0$$

and so

$$\omega_2(\beta_1 + \gamma_1) = \omega_1(\beta_1) = \omega_1(\beta_2) = \omega_2(\beta_2 + \gamma_2).$$

Hence, $\omega_2$ is well-defined.

# Differentials in extensions

**Proof.**

Since $\omega_1$ is K-linear so is $\omega_2$. Lemma 1 then implies that

$$\exists! \omega' : \mathbb{A}_F \to L \quad \text{s.t.} \quad \mathrm{Tr}_{L/K} \circ \omega' = \omega_2.$$

Now, for every $\beta \in \mathbb{A}_{F/E}$ we have

$$\begin{aligned} \mathrm{Tr}_{L/K}(\omega'(\beta)) &= \omega_2(\beta) \\ &= \omega_2(\beta + 0) \\ &= \omega_1(\beta) \\ &= \omega(\mathrm{Tr}_{F/E}(\beta)). \end{aligned}$$

# Differentials in extensions

**Proof.**

We turn to prove that $\omega'$ is a differential. To this end, we will show that $\omega'$ vanishes on $\Lambda_{\mathsf{F}}(D) + \mathsf{F}$.

Otherwise, since $\omega' : \mathbb{A}_{\mathsf{F}} \to \mathsf{L}$ is L-linear we will have that

$$\omega'(\Lambda_{\mathsf{F}}(D) + \mathsf{F}) = \mathsf{L}.$$

As $\mathrm{Tr}_{\mathsf{L}/\mathsf{K}}$ is onto K, we have that

$$\mathrm{Tr}_{\mathsf{L}/\mathsf{K}}(\omega'(\Lambda_{\mathsf{F}}(D) + \mathsf{F})) = \mathsf{K}.$$

But recall that $\mathrm{Tr}_{\mathsf{L}/\mathsf{K}} \circ \omega' = \omega_2$ and so

$$\omega_2(\Lambda_{\mathsf{F}}(D) + \mathsf{F}) = \mathsf{K}.$$

# Differentials in extensions

**Proof.**

$$\omega_2(\Lambda_{\mathsf{F}}(D) + \mathsf{F}) = \mathsf{K}. \tag{1}$$

Recall that every element of $\mathbb{A}_{\mathsf{F}}$ can be written as $\beta + \gamma$ where $\beta \in \mathbb{A}_{\mathsf{F}/\mathsf{E}}$ and $\gamma \in \Lambda_{\mathsf{F}}(D)$, and that we defined

$$\omega_2(\beta + \gamma) = \omega_1(\beta).$$

Thus,

$$\omega_2(\Lambda_{\mathsf{F}}(D)) = \omega_1(0) = 0. \tag{2}$$

Further, by Lemma 5,

$$\omega_1(\Lambda_{\mathsf{F}/\mathsf{E}}(D) + \mathsf{F}) = 0 \quad \Longrightarrow \quad \omega_2(\mathsf{F}) = \omega_1(\mathsf{F}) = 0 \tag{3}$$

Equations (2),(3) imply

$$\omega_1(\Lambda_{\mathsf{F}}(D) + \mathsf{F}) = 0,$$

in contradiction to Equation (1).

# Differentials in extensions

**Proof.**

We turn to establish uniqueness. Assume there is $\omega'' : \mathbb{A}_\mathsf{F} \to \mathsf{L}$ s.t.

$$\forall \beta \in \mathbb{A}_{\mathsf{F}/\mathsf{E}} \qquad \mathsf{Tr}_{\mathsf{L}/\mathsf{K}}(\omega''(\beta)) = \mathsf{Tr}_{\mathsf{L}/\mathsf{K}}(\omega'(\beta)) = \omega(\mathsf{Tr}_{\mathsf{F}/\mathsf{E}}(\beta)).$$

Then, $\eta = \omega'' - \omega'$ is a differential of F/L and, in particular is L-linear. Hence,

$$\mathsf{Tr}_{\mathsf{L}/\mathsf{K}}(\eta(\beta)) = \mathsf{Tr}_{\mathsf{L}/\mathsf{K}}(\omega''(\beta)) - \mathsf{Tr}_{\mathsf{L}/\mathsf{K}}(\omega'(\beta)) = 0.$$

As $\mathsf{Tr}_{\mathsf{L}/\mathsf{K}}$ is onto, we have that

$$\eta(\mathbb{A}_{\mathsf{F}/\mathsf{E}}) \subsetneq \mathsf{L}.$$

By the L-linearity of $\eta$, we get that $\eta(\mathbb{A}_{\mathsf{F}/\mathsf{E}}) = 0$.

Since $\eta$ is a differential it also vanishes on some $\Lambda_\mathsf{F}(D')$ for some divisor $D'$ and so, by Lemma 4, $\eta$ vanishes on $\mathbb{A}_\mathsf{F}$, namely, $\omega' = \omega''$.

# Differentials in extensions

**Proof.**

To conclude the proof, we show that

$$(\omega') = D = \mathrm{Con}_{\mathsf{F}/\mathsf{E}}((\omega)) + \mathrm{Diff}(\mathsf{F}/\mathsf{E}).$$

We already proved that $\omega'$ vanishes on $D$, and so we need to prove that $D$ is the largest such divisor.

To this end, take $D' \in \mathcal{D}(\mathsf{F})$ s.t. $D' \not\leq D$. We will show that

$$\exists \beta \in \Lambda_{\mathsf{F}}(D') \quad \text{s.t.} \quad \omega'(\beta) \neq 0.$$

By Lemma 7,

$$\exists \beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(D') \subseteq \Lambda_{\mathsf{F}}(D') \quad \text{s.t.} \quad \omega_1(\beta) \neq 0.$$

However, $\beta \in \Lambda_{\mathsf{F}/\mathsf{E}}(D')$ and so $\omega_2(\beta) = \omega_1(\beta) \neq 0$.

As $\omega_2(\beta) = \mathrm{Tr}_{\mathsf{L}/\mathsf{K}}(\omega'(\beta))$ we conclude that $\omega'(\beta) \neq 0$. □

# Overview

# The co-trace

## Definition 9

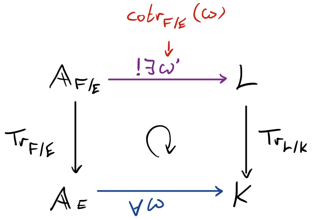
The map

$$\mathrm{cotr}_{F/E} : \Omega_{E/K} \to \Omega_{F/L}$$
$$\omega \mapsto \omega'$$

that is defined implicitly by the property

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega) = \omega \circ \mathrm{Tr}_{F/E}$$

on $\mathbb{A}_{F/E}$ is called the co-trace.



# Differential

**Claim 10**

Let $\omega_1, \omega_2 \in \Omega_{E/K}$. Then,

$$\mathrm{cotr}_{F/E}(\omega_1 + \omega_2) = \mathrm{cotr}_{F/E}(\omega_1) + \mathrm{cotr}_{F/E}(\omega_2).$$

**Proof.**

We have that

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_1) = \omega_1 \circ \mathrm{Tr}_{F/E},$$
$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_2) = \omega_2 \circ \mathrm{Tr}_{F/E}.$$

Thus,

$$\mathrm{Tr}_{L/K} \circ (\mathrm{cotr}_{F/E}(\omega_1) + \mathrm{cotr}_{F/E}(\omega_2)) =$$
$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_1) + \mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega_2) =$$
$$\omega_1 \circ \mathrm{Tr}_{F/E} + \omega_2 \circ \mathrm{Tr}_{F/E} = (\omega_1 + \omega_2) \circ \mathrm{Tr}_{F/E},$$

and the proof follows by the (implicit) definition of $\mathrm{cotr}_{F/E}(\omega_1 + \omega_2)$.

# Differential

**Claim 11**

Let $\omega \in \Omega_{E/K}$, and $x \in E$. Then,

$$\mathrm{cotr}_{F/E}(x\omega) = x\mathrm{cotr}_{F/E}(\omega).$$

**Proof.**

Let $[x]$ be the function on $\mathbb{A}_{F/E}$ mapping $\alpha \mapsto x\alpha$. By the implicit definition of $\mathrm{cotr}_{F/E}$ we have that on $\mathbb{A}_{F/E}$,

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega) \circ [x] = \omega \circ \mathrm{Tr}_{F/E} \circ [x].$$

Now, for every $\alpha \in \mathbb{A}_{F/E}$,

$$(\mathrm{Tr}_{F/E} \circ [x])(\alpha) = \mathrm{Tr}_{F/E}(x\alpha) = x\mathrm{Tr}_{F/E}(\alpha) = ([x] \circ \mathrm{Tr}_{F/E})(\alpha)$$

and so $\mathrm{Tr}_{F/E} \circ [x] = [x] \circ \mathrm{Tr}_{F/E}$ on $\mathbb{A}_{F/E}$. Thus,

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega) \circ [x] = \omega \circ [x] \circ \mathrm{Tr}_{F/E}.$$

# Differential

**Proof.**

$$\mathrm{Tr}_{L/K} \circ \mathrm{cotr}_{F/E}(\omega) \circ [x] = \omega \circ [x] \circ \mathrm{Tr}_{F/E}.$$

But

$$x\omega = \omega \circ [x],$$
$$x\mathrm{cotr}_{F/E}(\omega) = \mathrm{cotr}_{F/E}(\omega) \circ [x],$$

and so

$$\mathrm{Tr}_{L/K} \circ (x\mathrm{cotr}_{F/E}(\omega)) = (x\omega) \circ \mathrm{Tr}_{F/E}.$$

The proof follows by the (implicit) definition of $\mathrm{cotr}_{F/E}$. □

# Differential

**Claim 12**

Let $F/E$ and $F'/F$ be finite separable extensions of function fields. Then,

$$\mathrm{cotr}_{F'/E} = \mathrm{cotr}_{F'/F} \circ \mathrm{cotr}_{F/E}.$$

As with all tower type statement, we omit the proof.

# Overview

# Hurtwiz Genus Formula

**Theorem 13**

*Let* F/L *be a finite separable extension of* E/K*. Let* $g_E, g_F$ *be the corresponding genera. Then,*

$$2g_F - 2 = \frac{[F : E]}{[L : K]} \cdot (2g_E - 2) + \deg \mathrm{Diff}(F/E).$$

# Hurtwiz Genus Formula

$$2g_F - 2 = \frac{[F : E]}{[L : K]} \cdot (2g_E - 2) + \deg \mathrm{Diff}(F/E).$$

**Proof.**

Take $0 \neq \omega \in \Omega_{E/K}$. By Theorem 8,

$$(\mathrm{cotr}_{F/E}(\omega)) = \mathrm{Con}_{F/E}((\omega)) + \mathrm{Diff}(F/E).$$

As $(\omega), (\mathrm{cotr}_{F/E}(\omega))$ are canonical divisors of $E/K$ and $F/L$, respectively, Riemann-Roch theorem implies that

$$\deg(\omega) = 2g_E - 2 \qquad \deg_F(\mathrm{cotr}_{F/E}(\omega)) = 2g_F - 2.$$

The proof then follows as

$$\deg_F(\mathrm{Con}_{F/E}((\omega))) = \frac{[F : E]}{[L : K]} \cdot \deg_E(\omega).$$

□