Exercise 2: Single Qubit (and Two Qubits)

2.1 Single Qubit

1. We would like to understand phases of states (scalar multiples of states $e^{\theta i}$, for $\theta \in \mathbb{R}$).

   (a) Prove that if $|v\rangle$ is a qudit[1] quantum state that is a scalar multiple of another quantum state $|u\rangle$, $|v\rangle = \alpha |u\rangle$, then $\alpha = e^{\theta i}$ for some $\theta \in \mathbb{R}$. We say they are different only by some global phase.

   (b) We are given two states of a qudit $|u\rangle, |v\rangle$ as in the previous subquestion. Prove that for any choice of orthonormal basis, the measurement outcome distributions agree on $|u\rangle$ and $|v\rangle$.

   (c) Using the previous subquestion argue that performing any quantum operation on either $|u\rangle$ or $|v\rangle$ would have identical observable outcomes (the things we get answers to in the lab, i.e. the answers to measurements when we perform them as part of the operations).

   (d) Prove or disprove the following statement: As a result of the previous subqeustion, a qubit in the state $\frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$ and a qubit in the state $\frac{1}{\sqrt{2}}(|u_1\rangle + e^{\theta i} |u_2\rangle)$ for some $\theta \in \mathbb{R}$ and orthonormal $|u_1\rangle, |u_2\rangle$ would have the same observable outcomes in any quantum operation. We call the scalar factor multiplying $|u_2\rangle$ in both states a relative phase.

2. Light has a property called polarisation. We can think of polarisation as a quantum property of photons that is very appropriate for a qubit, as it is a direction in the 2-dimensional plane that is perpendicular to the photon's trajectory (see Figure 2.1). Polarisers are filters that pass light polarised in a specific direction in the filter's plane and blocks light that is polarised in the direction perpendicular to it. So polarisers work similarly to the devices we saw in the first lecture, where the "up" direction is thrown away (blocked). We got a new office that has on its only window a weird

[1] A qudit lives in d-dimensions for a $d \in \mathbb{N}$, rather than just 2 as in qubits. We sometimes call these dimesions "levels".

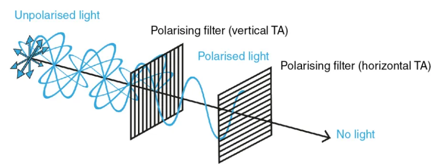

Figure 2.1: Polarisers and polarisation of light[2]

apparatus: two polarisers, one in a vertical alignment and another after it, in a horizontal alignment. This means we get no light into our office!

(a) Prove that the probability a photon goes through both polarisers is 0.

(b) Fortunately there is quite a big gap between the polarisers in our window, and we found polarisers in the office's drawer. We would like to take advantage of that in order to get some light in. We will introduce a new polariser between the fixed ones. The new polariser's direction will be a parameter θ. Calculate the probability of a photon that passes through the first polariser to pass through both the new polariser and the final (horizontal) one as a function of θ. What is the best θ to maximise this probability and its corresponding maximal probability?

(c) Is the maximum we calculated in the previous subquestion the best we can do without removing the original polarisers?

3. Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle = \beta_+ |+\rangle + \beta_- |-\rangle$ be a state of a qubit. Define its "bit uncertainty" as $S(|\psi\rangle) := |\alpha_0| + |\alpha_1|$ and similarly its "sign uncertainty" as $\hat{S}(|\psi\rangle) := |\beta_+| + |\beta_-|$. These are measures of how unsure we are regarding the outcome of the state in a measurement in the corresponding basis. In this question we will see a version/analogue of the Uncertainty Principle: we cannot be sure of both bases simultaneously.

(a) Calculate the uncertainties for the computational basis and for the Hadamard basis of a qubit, i.e. $S(|\psi\rangle), \hat{S}(|\psi\rangle)$ for $|\psi\rangle \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$.

(b) What is the range of S and of $\hat{S}$? (what are the possible evaluations of them?) When do they achieve the endpoints of that range?

(c) Show that the minimum of the function $f(x) := x^2 + x\sqrt{1-x^2}$ for $x \in [\frac{1}{\sqrt{2}}, 1]$ is 1[3].

(d) Prove a lower bound for the product of the uncertainties $S(|\psi\rangle) \cdot \hat{S}(|\psi\rangle)$ that is strictly larger than 1.

[2] credit: `https://entokey.com/polarisation/`

[3] The red font signifies a fix in the exercise

(e) Conclude from the previous subquestion a lower bound for $\max\{S(|\psi\rangle), \hat{S}(|\psi\rangle)\}$, showing that at least one property must be uncertain (i.e. at least one distribution in the corresponding basis must have support on both options).

4. We would like to distinguish between two specific qubit states $|u\rangle, |v\rangle$ in the sense that given $|\psi\rangle \in \{|u\rangle, |v\rangle\}$ we are tasked to pronounce which one we were given. Let's understand how well we can perform this task. For convenience, assume $|u\rangle, |v\rangle$ have real amplitudes.

   (a) Prove the only important factor here is the angle between the two vectors, $\langle u|v\rangle$.

   (b) Two-sided error: using the previous subquestion, we may decide to orient the vectors $|u\rangle, |v\rangle$ in the real plane, where the the bisector (the mid-angle) is at $|+\rangle$, where $|u\rangle$ is closer to $|0\rangle$ and $|v\rangle$ to $|1\rangle$. Then we can propose to measure in the computational basis and guess that if we got the outcome $|0\rangle$ we started with $|u\rangle$, and if we got the outcome $|1\rangle$ we started with $|v\rangle$. What kind of errors are possible in our protocol? Calculate the probability of an error in this protocol.

   (c) One-sided error: propose a different protocol, where the error is asymmetric. We want to be infallible, meaning having no probability to err, if we get $|u\rangle$, and for that we allow ourselves to make mistakes if we are given $|v\rangle$, but hopefully with as little probability as possible. What is the error probability you can achieve, in case we get $|v\rangle$?

   (d) "Zero-sided error": of course the previous subquestion could easily be made into a protocol that can err only if we are given $|v\rangle$. Use these two one-sided error protocols, one for each side, and construct a protocol that is allowed to guess also a don't-know answer, i.e. one of the three $\{|u\rangle, |v\rangle, \text{"don't know"}\}$, but when it guesses a vector it must be correct. Clearly we would have liked the probability of "don't know" to be minimal. What is the probability your procedure achieves?

2.2 Two Qubits

1. A state $|\psi\rangle$ is *separable* (or an *unentangled state*) if $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ for some states $|\phi_1\rangle, |\phi_2\rangle$. A state that is not separable is *entangled*. Which of the following states are separable? Prove your answers.

   (a) $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$

   (b) $\frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}$

   (c) $\frac{|00\rangle + i|01\rangle + i|10\rangle - |11\rangle}{2}$

(d) $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$