Recitation 5: The Ramification and Residual Indices

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Let F be a field and $\nu\colon F \to \Gamma \cup \{\infty\}$ a valuation. Recall that its corresponding valuation ring is

$$\mathcal{O}_F = \{a \in F \mid \nu(a) \geq 0\}.$$

This is a local ring with maximal ideal

$$\mathfrak{m}_F = \{a \in F \mid \nu(a) > 0\}.$$

The quotient $\overline{F} := \mathcal{O}_F/\mathfrak{m}_F$ is a field (called the *residue field*), and the map $\varphi\colon F \to \overline{F} \cup \{\infty\}$ given by

$$\varphi(f) = \begin{cases} f + \mathfrak{m}_F & f \in \mathcal{O}_F \\ \infty & \text{otherwise} \end{cases}$$

is a corresponding place.

You showed in class that if $E \subseteq F$ is a subfield, then the restriction $\nu|_E\colon E \to \Gamma \cup \{\infty\}$ is a valuation. Its valuation ring is $\mathcal{O}_E = E \cap \mathcal{O}_F$ and its maximal ideal is $\mathfrak{m}_E = E \cap \mathfrak{m}_F$. Moreover, the restriction $\varphi|_E$ is a place of E (corresponding to the valuation $\nu|_E$). Its residue field is

$$\overline{E} = \varphi(E) \backslash \{\infty\} \cong \mathcal{O}_E/\mathfrak{m}_E$$

and is a subfield of $\overline{F}$.

Definition 1. The *ramification index* of F/E is $(\nu(F^\times) : \nu(E^\times))$.

Definition 2. The *residual index* of F/E is $[\overline{F} : \overline{E}]$.

Theorem 3.

$$[\overline{F} : \overline{E}] \cdot (\nu(F^\times) : \nu(E^\times)) \leq [F : E].$$

Corollary 4. *If $[E : F] = n < \infty$ then both $(\nu(F^\times) : \nu(E^\times)) \leq n$ and $[\overline{F} : \overline{E}] \leq n$.*

Proof of Theorem 3. For $z \in \mathcal{O}_F$, let $\overline{z} := \varphi(z) = z + \mathfrak{m}_F \in \overline{F}$. Let $x_1, \dots, x_m \in \mathcal{O}_F$ be such that $\overline{x_1}, \dots, \overline{x_m} \in \overline{F}$ are linearly independent over $\overline{E}$. Let $y_1, \dots, y_n \in F^\times$ be such that $\nu(y_1), \dots, \nu(y_n)$ represent distinct cosets in the quotient group $\nu(F^\times)/\nu(G^\times)$.

It suffices to prove that the subset $\{x_i y_j\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \subseteq F$ is linearly independent over E. Suppose

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j = 0 \tag{1}$$

for $a_{ij} \in E$ which are not all zero. W.l.o.g. for every $j \in [n]$ there exists $i \in [m]$ such that $a_{ij} \neq 0$ (otherwise omit j in the summation).

Claim. $\nu\left(\sum_{i=1}^m a_{ij}x_i\right) = \min_i \nu(a_{ij})$. *In particular,* $\nu\left(\sum_{i=1}^m a_{ij}x_i\right) \in \nu(E^\times)$.

Indeed, let $k \in [m]$ be such that $\nu(a_{kj}) = \min_i \nu(a_{ij})$. By the assumption, $a_{kj} \neq 0$. We need to show that $\nu\left(\sum_{i=1}^m a_{ij}x_i\right) = \nu(a_{kj})$. Let $b_{ij} := \frac{a_{ij}}{a_{kj}}$ so that $b_{kj} = 1$ and

$$\nu(b_{ij}) = \nu(a_{ij}) - \nu(a_{kj}) \geq 0.$$

Then $b_{ij} \in \mathcal{O}_E$ for all i and $b_{kj} \notin \mathfrak{m}_E$, hence $\overline{b_{ij}} \in \overline{E}$ and $\overline{b_{kj}} \neq 0$. Since $\overline{x_1}, \ldots, \overline{x_m} \in \overline{F}$ are linearly independent over $\overline{E}$,

$$\overline{\sum_{i=1}^m b_{ij}x_i} = \sum_{i=1}^m \overline{b_{ij}}\,\overline{x_i} \neq 0.$$

It follows that $\sum_{i=1}^m b_{ij}x_i \in \mathcal{O}_F \backslash \mathfrak{m}_F$, hence $\nu\left(\sum_{i=1}^m b_{ij}x_i\right) = 0$. Therefore,

$$\nu\left(\sum_{i=1}^m a_{ij}x_i\right) = \nu\left(a_{kj}\sum_{i=1}^m b_{ij}x_i\right) = \nu(a_{kj}) + \nu\left(\sum_{i=1}^m b_{ij}x_i\right) = \nu(a_{kj})$$

as desired.

To conclude, by Equation (1) we have

$$\sum_{j=1}^n \left(\sum_{i=1}^m a_{ij}x_i\right) y_j = 0.$$

By the claim, for each j we have $\sum_{i=1}^m a_{ij}x_i \neq 0$ (and clearly $y_j \neq 0$). Hence $n \geq 2$. This implies that there exist $k \neq \ell$ such that

$$\nu\left(\left(\sum_{i=1}^m a_{ik}x_i\right) y_k\right) = \nu\left(\left(\sum_{i=1}^m a_{i\ell}x_i\right) y_\ell\right),$$

i.e.

$$\nu\left(\sum_{i=1}^m a_{ik}x_i\right) + \nu(y_k) = \nu\left(\sum_{i=1}^m a_{i\ell}x_i\right) + \nu(y_\ell).$$

But then

$$\nu(y_k) - \nu(y_\ell) = \nu\left(\sum_{i=1}^m a_{i\ell}x_\ell\right) - \nu\left(\sum_{i=1}^m a_{ik}x_i\right) \in \nu(E^\times),$$

contradicting the fact that $\nu(y_k)$ and $\nu(y_\ell)$ are in different cosets in $\nu(F^\times)/\nu(E^\times)$. □