On Raz and Reingold PRG and A New White Box WPRG, and Error Reduction For Weighted PRGs Against Read Once Branching Programs

Master's Thesis Presentation

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BPL vs. L

Definition (Space Bounded Classes)

1. L is the class of all languages decidable by logarithmic space TM,
2. BPL, RL defined as all languages that are decidable by logarithmic space probabilistic TM with *two* or *one* sided error, respectively.

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The problem

What are the relations between BPL, RL and L?

Read Once Branching Programs

The uniform PTM model is difficult to work with directly.

Lemma

Every PTM with fixed input length n that uses space $s(n)$ can be represented by an $(n, w = 2^{O(s(n))})$ Read Once Branching Program (ROBP, BP).

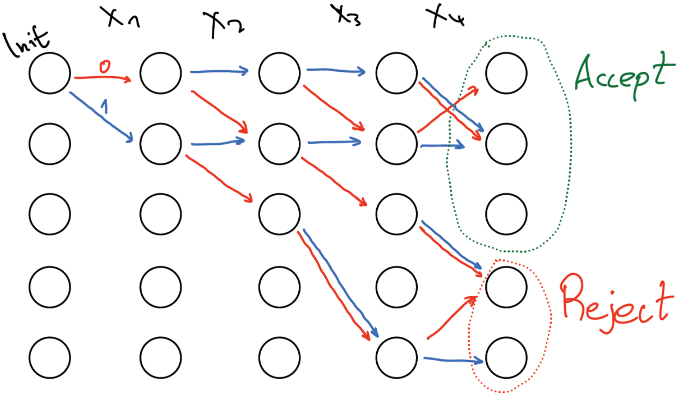

Figure: An $(n = 4, w = 5)$ BP.

Approaches for Derandomization

Goal: approximate acceptance probability of all (n, w) machines.

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Definition (Der. Algorithm)

Given an (n, w) BP, approximate its acceptance probability up to error ε.

This is a **White Box** technique, as its allowed to *inspect* the input.

Approaches for Derandomization (cont.)

Another approach is in oblivious manner, i.e. **Black Box** techniques.

Definition (PRG)

A function $G : \{0,1\}^s \to \{0,1\}^n$ is called an (n, w, ε) PRG, if for every (n, w) BP M,

$$|\mathbb{E}[M(U_n)] - \mathbb{E}[M(G(U_s))]| \leq \varepsilon.$$

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A pair of functions $(G, \mu) : \{0,1\}^s \to \{0,1\}^n \times \mathbb{R}$ are called (n, w, ε) Weighted-PRG, if for every (n, w) BP M,

$$\left| \mathbb{E}[M(U_n)] - \mathop{\mathbb{E}}_{x \sim U_s} [\mu(x) \cdot M(G(x))] \right| \leq \varepsilon.$$

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For every machine M, the Derandomization via PRG/WPRG (G, μ) follows as:

1. Enumerate seeds $x \in \{0,1\}^s$,
2. Compute $G(x)$,
3. Avg. the result of $M(G(x))$,
   - For WPRG: consider the *weighted* avg.

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$\Longrightarrow$ Computing $\mathbb{E}[M(U_n)] \pm \varepsilon$ takes space: $space(G) + seed(G)$

Brief History of Derandomization

Space	Ref
$O(\lg n \cdot \lg \frac{nw}{\varepsilon})$	Sav 70', BCP 83', Nis 92
$O(\sqrt{\lg n} \cdot \lg \frac{nw}{\varepsilon})$	SZ 95'
$O(\sqrt{\lg n} \cdot \lg(nw) + \lg\lg_{nw} \frac{1}{\varepsilon})$	AKMPVS 21
$O(\sqrt{\lg n} \cdot \lg(\frac{nw}{\varepsilon}) \cdot \frac{1}{\sqrt{\lg\lg n}})$	Hoza 21

Der. of (n,w) BP

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$O\left(\frac{\lg n \cdot \lg \frac{nw}{\varepsilon}}{\max\{1, \lg\lg w - \lg\lg \frac{n}{\varepsilon}\}}\right)$	PRG	Arm 98'
$\tilde{O}(\lg n \cdot \lg(nw) + \lg \frac{1}{\varepsilon})$	WPRG	BCG 18', CL20', CDRSTS 21', PV 21'
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PRG & WPRG for (n,w) BP

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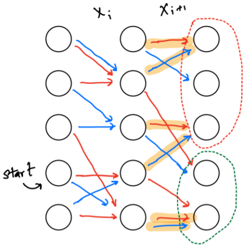
One step. Prove the Recycle lemma for arbitrary $w \in \mathbb{N}$ and $\varepsilon_{\mathcal{P}} > 0$:

Recycle Lemma

$\exists$ an explicit construction of pseudo random family $\mathcal{P}$ s.t. for every $(2, w)$ BP M,

$$p \sim U_{\mathcal{P}} \implies M_p \approx_{\varepsilon_{\mathcal{P}}} M^2$$



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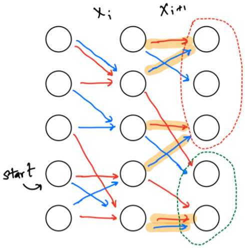

Deduce PRG recursively.
Sample *independently* $\overline{p} = (p_1, \ldots, p_h)$ so for every $(2^h, w)$ BP M,

$$M_{p_1,\ldots,p_h} \approx_{\varepsilon(1)} M^2_{p_1,\ldots,p_{h-1}} \approx_{\varepsilon(2)} \ldots \approx_{\varepsilon(h)} M^{2^h}$$

Rough recipe for PRGs by [Nis92; INW94] (cont.)

Analysis. The family size in both constructions is

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$$\varepsilon(h) = 2\varepsilon(h-1) + \varepsilon_{\mathcal{P}}.$$

Thus $\varepsilon \stackrel{\text{def}}{=} \varepsilon(\log n) = n \cdot \varepsilon_{\mathcal{P}}$, so the seed length becomes

$$\begin{aligned} s &= \log n \cdot (\log |\mathcal{P}|) \\ &= O(\log n \cdot (\log w + \log \varepsilon_{\mathcal{P}}^{-1})) \\ &= O(\log n \cdot (\log n + \log w + \log \varepsilon^{-1})) \end{aligned}$$

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Gil's Road map to space bounded computation

Improving Nisan [Nis92] PRG.
Better analysis of the error may lead to PRG with seed length

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Improving Saks and Zhou [SZ99] derandomization.

1. The WPRG of Braverman, Cohen, and Garg [BCG18] already has seed length

$$\mathsf{s}_{\mathsf{BCG}} = \widetilde{O}(\log n \cdot (\log n + \log w) + \log \varepsilon^{-1}),$$

2. while Raz and Reingold [RR99] obtained *conditional* PRG with seed length

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So maybe combine somehow [BCG18] and [RR99] to get

$$s_{\text{hopefully}} = \widetilde{O}(\log n \cdot (\log n) + \log w + \log \varepsilon^{-1}).$$

Plugging such a good seeded PRG into [SZ99] framework would yield $\mathsf{BPL} \subseteq \mathsf{L}^{4/3}$.

Contribution summary

Theorem ([CDR+21; PV21])

Let $G_0 : \{0,1\}^{s_0} \to \{0,1\}^n$ *be an* $(n, w, \varepsilon_0 = 1/n^2)$ *Black Box PRG. Then, for every error parameter* $0 < \varepsilon < \varepsilon_0$ *there exists an* (n, w, ε) *Black Box WPRG with seed length*

$$s_0 + O\left((\log w + \log \varepsilon^{-1}) \cdot \log\log_n(1/\varepsilon)\right)$$

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Brief exposition of Extractors

Definition

A distribution $X \subseteq \{0,1\}^{n}$ is called (n, k) **source** if

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Theorem (Lower bound)

A (k,ε) extractor is **optimal** *if*

$$d = O(\log(n/\varepsilon))$$
$$m = k + d - O(\log \varepsilon^{-1}).$$

For simplicity we assume such extractors are fully explicit (i.e. computable in linear space)...

Toy example of [INW94]

We focus on fooling $(n+m, w)$ BPs.

1. Let Ext be a $(k_{\mathrm{INW}} = n - (\log w - \log \varepsilon^{-1} - 1), \varepsilon/2)$ *optimal* extractor, where

$$\mathrm{Ext} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m.$$

2. The PRG $\mathrm{INW} : \{0,1\}^{n+d} \to \{0,1\}^{n+m}$ is defined as

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Claim

INW is an ε-PRG, i.e. for every $(n+m, w)$ BP M,

$$|\Pr[M(U_{n+m}) \text{ acc}] - \Pr[M(\mathrm{INW}(U_{n+d})) \text{ acc}]| \leq \varepsilon.$$

Toy example of [INW94] (cont.)

INW PRG

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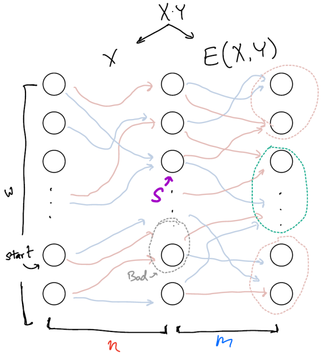
Analysis.
Let $X \circ Y \sim U_{n+d}$, and M be some BP. Let $s \sim M(X)$.



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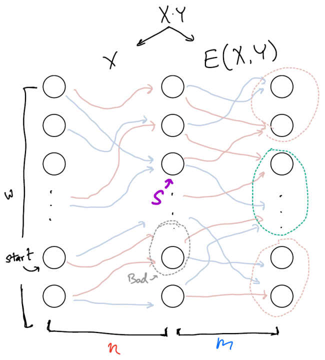

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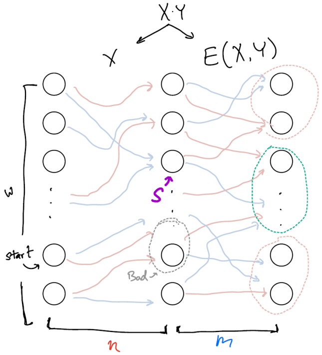

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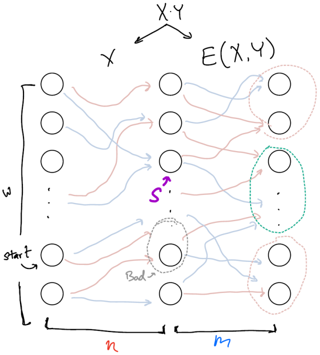
Define X_s as the uniform dist. over $M^{-1}(s)$.
It is not hard to prove:

$$s \notin \mathrm{Bad} \implies X_s \text{ is an } (n, k_{\mathrm{INW}}) \text{ source.}$$



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Thus,

$$\begin{aligned}
&|\Pr[M(U_{n+m}) \text{ acc}] - \Pr[M(\mathrm{INW}(U_{n+d})) \text{ acc}]| \\
&\quad = |\Pr[M(X \circ U_m) \text{ acc}] - \Pr[M(\mathrm{INW}(X, Y)) \text{ acc}]| \\
&\quad = \left| \sum_{s \in [w]} \Pr[M(X) = s] \cdot (\Pr[M_s(U_m) \text{ acc}] - \Pr[M_s(\mathrm{Ext}(X_s, Y)) \text{ acc}]) \right| \\
&\quad \le \left| \sum_{s \notin \mathrm{Bad}} \Pr[M(X) = s] \cdot \mathsf{SD}(U_m, \mathrm{Ext}(X_s, Y)) \right| + \left| \sum_{s \in \mathrm{Bad}} \Pr[M(X) = s] \cdot 1 \right| \\
&\quad \le 1 \cdot \frac{\varepsilon}{2} + w \cdot \frac{\varepsilon}{2w} \\
&\quad \le \varepsilon.
\end{aligned}$$

Full construction of [INW94]

The PRG $\mathrm{INW}_{h+1} : \{0,1\}^{\ell_{h+1}} \times \{0,1\}^{d_{h+1}} \to \{0,1\}^{m_h}$ defined as

$$\mathrm{INW}_{h+1}(x \circ y) \stackrel{\text{def}}{=} \mathrm{INW}(x) \circ \mathrm{INW}(\mathrm{Ext}_h(x,y))$$
$$\mathrm{INW}_1(x) \stackrel{\text{def}}{=} x_0$$

where $\mathrm{Ext}_{h+1} : \{0,1\}^{\ell_{h+1}} \times \{0,1\}^{d_{h+1}} \to \{0,1\}^{m_h}$ is a $(k_{h+1}, \varepsilon_{\mathrm{Ext}})$ extractor.

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so $\varepsilon = \varepsilon(\log n) = n \cdot \varepsilon_{\mathrm{Ext}}$ forces seed length

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   1. PRGs are much more powerful than derandomizations! (as Saks and Zhou [SZ99] showed us)
   2. We combine solution (1) with recent developments to conclude a new white box Weighted PRG

Derandomization implies Estimators

Let A be an $(n, w, \varepsilon_{\mathbf{A}})$ (additive) derandomization, M be an (n, w) BP with two states s, t.
We wish to compute $\widetilde{p_{s,t}}$ s.t.

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While potentially $p_{s,t} = 2^{-\Theta(n)}$, its easy to discard such low probabilities **in the analysis**.

PRG Analysis

Say we wish to construct an $(n, w, \boldsymbol{\varepsilon_G})$ PRG G.
By setting $\boldsymbol{\varepsilon_G} \leftarrow \boldsymbol{\varepsilon_G} + \gamma \cdot nw$, one may assume that **always**

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Thus,

$$r = O(\varepsilon_{\mathrm{A}}/\gamma).$$

Parameters. Set $\gamma = 1/nw$ and let A be [SZ99] with $\varepsilon_{\mathrm{A}} = 1/nw \quad \Longrightarrow \quad r = O(1)$.

A New PRG

First, we conclude an explicit white box PRG from [RR99] using [SZ99] as an estimator:

Theorem (PRG based [RR99])

There exists an (n, w, ε) *white box PRG with seed length*

$$\mathsf{s}_{\mathsf{RR}} = \widetilde{O}(\log n \cdot (\log n + \log \varepsilon^{-1}) + \log w),$$

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Now use the black box error reduction of [CDR+21; PV21] to conclude:**

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Notations:

1. For any event E, we define $\mathbf{H}(E) = \log \frac{1}{\Pr[E]}$.
2. For every $q \in [n]$ and $s \in [w]$,

$$S^{\mathrm{ideal,q}}(s) \stackrel{\mathsf{def}}{=} \Pr[M_{s_{\mathrm{init}}}(U_q) = s].$$

3. For simplicity, assume A is an $(n, w, r = 0)$ **perfect estimator**.
4. The estimated entropy of a given state is

$$\widetilde{\mathbf{H}}(s) \stackrel{\mathsf{def}}{=} \log \frac{1}{\mathrm{A}(M, s_{\mathrm{init}}, s)},$$

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The PRG

Begin with the INW PRG, and we modify it gradually:

$$G(x, y) \stackrel{\text{def}}{=} G^{\mathrm{A}}(x, y) \stackrel{\text{def}}{=} x \circ \mathrm{Ext}(x, y).$$

Improving the entropy loss

Analysis.
Let $X \circ Y \sim U_{n+d}$. Let $S_{\text{mid}}^{\text{ideal}}, S_{\text{mid}}^{\text{gen}}$ be state distribution after walking via U_n or $G^{\text{A}}(X)$.

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Improving the entropy loss (cont.)

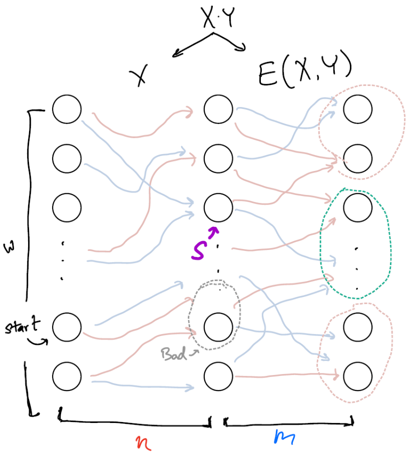

Figure: INW

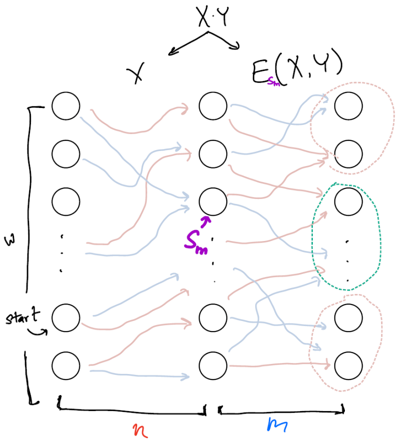

Figure: G^{A}

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So we summarize:

1. $k_{s_{\text{mid}}} = n - \widetilde{\mathbf{H}}(s_{\text{mid}})$,
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Using the same trick as before, we discard all states that satisfies

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Now we can bound for every $s_{\text{mid}} \in \text{supp}(S^{\text{gen}}_{\text{mid}})$:

$$k_{s_{\text{mid}}} = n - \widetilde{\mathbf{H}}(s_{\text{mid}}) \geq n - \log w - \log \varepsilon^{-1} - 1 - \log n - \log m.$$

so we gained nothing...

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The input length

Let $s \in [nw]$. Define $\ell_s \stackrel{\text{def}}{=} n - \widetilde{\mathbf{H}}(s)$.
The input source X is redefined to have length $\ell_{S_{\mathrm{in}}^{\mathrm{ideal}}}$, i.e. *random variable*.

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$\mathbf{H}_\infty(S_{\text{in}}^{\text{ideal}} \circ X) = n$

We circumvent the min entropy definition and use $\mathbf{H}$ instead. For every $s_{\text{in}} \circ x$,

$$\begin{aligned}\mathbf{H}(S_{\text{in}}^{\text{ideal}} \circ X = s_{\text{in}} \circ x) &= \mathbf{H}(S_{\text{in}}^{\text{ideal}} = s_{\text{in}}) + \ell_{s_{\text{in}}} \\ &= \mathbf{H}(S_{\text{in}}^{\text{ideal}} = s_{\text{in}}) + n - \widetilde{\mathbf{H}}(s_{\text{in}}) \\ &= n.\end{aligned}$$

□

One step of recycling

Reminder

Let $s \in [nw]$. Define $\ell_s \stackrel{\text{def}}{=} n - \widetilde{\mathbf{H}}(s)$.

Let $X \circ Y \sim U_{\ell_{S_{\text{in}}^{\text{ideal}}}} \times U_d$. Let $S_{\text{mid}}^{\text{ideal}}, S_{\text{mid}}^{\text{gen}}$ be state distribution after walking via U_n or $G^{\text{A}}(X)$.

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$$\mathsf{SD}(S_{\text{mid}}^{\text{ideal}}, S_{\text{mid}}^{\text{gen}}) = 0.$$

Let $s_{\text{mid}} \sim S_{\text{mid}}^{\text{gen}}$.

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Let $s_{\text{mid}} \sim S_{\text{mid}}^{\text{gen}}$.
Analyzing the entropy. What is $\mathbf{H}_\infty(S_{\text{in}}^{\text{ideal}} \circ X \mid S_{\text{mid}}^{\text{gen}} = s_{\text{mid}})$?

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$\mathbf{H}_\infty(S_{\text{in}}^{\text{ideal}} \circ X \mid S_{\text{mid}}^{\text{gen}} = s_{\text{mid}}) = \ell_{s_{\text{mid}}}$

We again circumvent $\mathbf{H}_\infty$ with $\mathbf{H}$:

$$\begin{aligned}
&\mathbf{H}(S_{\text{in}}^{\text{ideal}} \circ X = s_{\text{in}} \circ x \mid S_{\text{mid}}^{\text{gen}} = s_{\text{mid}}) \\
&\quad \geq \mathbf{H}(S_{\text{in}}^{\text{ideal}} \circ X = s_{\text{in}} \circ x) - \mathbf{H}(S_{\text{mid}}^{\text{gen}} = s_{\text{mid}}) \\
&\quad = \mathbf{H}(S_{\text{in}}^{\text{ideal}} = s_{\text{in}}) + \ell_{s_{\text{in}}} - \mathbf{H}(S_{\text{mid}}^{\text{gen}} = s_{\text{mid}}) \\
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&\quad = \ell_{s_{\text{mid}}} + (\ell_{s_{\text{in}}} - \ell_{s_{\text{mid}}}) + \mathbf{H}(S_{\text{in}}^{\text{ideal}} = s_{\text{in}}) - \mathbf{H}(S_{\text{mid}}^{\text{ideal}} = s_{\text{mid}}) \\
&\quad = \ell_{s_{\text{mid}}} + (-\widetilde{\mathbf{H}}(s_{\text{in}}) + \widetilde{\mathbf{H}}(s_{\text{mid}})) + \mathbf{H}(S_{\text{in}}^{\text{ideal}} = s_{\text{in}}) - \mathbf{H}(S_{\text{mid}}^{\text{ideal}} = s_{\text{mid}}) \\
&\quad = \ell_{s_{\text{mid}}}
\end{aligned}$$

One step of recycling (cont.)

Reminder

Let $s \in [nw]$. Define $\ell_s \stackrel{\text{def}}{=} n - \widetilde{\mathbf{H}}(s)$.

We want to recycle $(S_{\text{in}}^{\text{ideal}} \circ X \mid S_{\text{mid}}^{\text{gen}} = s_{\text{mid}})$. We choose $(k_{s_{\text{mid}}} = \ell_{s_{\text{mid}}}, \varepsilon)$ extractor

$$\text{Ext}_{s_{\text{mid}}} : \{0,1\}^{n'_{S_{\text{in}}^{\text{ideal}}}} \times \{0,1\}^d \to \{0,1\}^m$$

where

$$n'_{S_{\text{in}}^{\text{ideal}}} = \log(nw) + \ell_{S_{\text{in}}^{\text{ideal}}}$$

Using optimal extractors,

$$m = k_{s_{\text{mid}}} - O(\log(1/\varepsilon)) = \ell_{s_{\text{mid}}} - O(\log(1/\varepsilon))$$
$$d = O(\log(n'_{S_{\text{in}}^{\text{ideal}}}/\varepsilon)) = O(\log n + \log(1/\varepsilon) + \log\log w)$$

One step of recycling (cont.)

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Thus, since the analysis holds for every s_{mid},

$$S_{\text{mid}}^{\text{gen}} \circ \text{Ext}(S_{\text{in}}^{\text{ideal}} \circ X \mid S_{\text{mid}}^{\text{gen}}, Y) \approx_\varepsilon S_{\text{mid}}^{\text{gen}} \circ U_{m_{S_{\text{mid}}^{\text{gen}}}}$$
$$= S_{\text{mid}}^{\text{ideal}} \circ U_{m_{S_{\text{mid}}^{\text{ideal}}}}$$
$$= S_{\text{mid}}^{\text{ideal}} \circ U_{\ell_{S_{\text{mid}}^{\text{ideal}}} - O(\log 1/\varepsilon)}$$

One step of recycling (cont.)

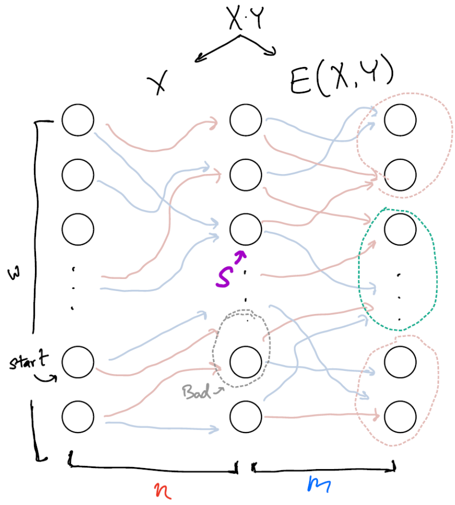

Figure: INW

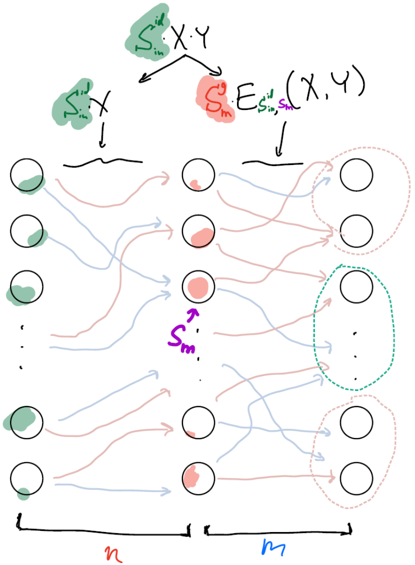

Figure: G^{A}

A full construction

In similar fashion to INW, we can devise (n, w, ε) white box PRG with seed length

$$\mathsf{s}_0 = O(\log n \cdot (\log n + \log \varepsilon^{-1} + \log \log w) + \log w)$$

but its space complexity is

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To solve the problems...

1. Save only 2 states instead of up to $\log n$ (which multiplied by $\times \log w$)
2. Use global buffers to maintain linear space
3. Use condensers to collect the extractors unavoidable loss

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Stochastic Matrix Powering and Derandomization

Let M be (n, w) BP.

1. By increasing the width $w \mapsto \mathrm{poly}(n, w)$, wlog all layers are identical.
2. Abuse notation to denote $\mathbf{M}$ the stochastic *transition matrix* of every layer.
3. Define $\mathbf{M} = \frac{1}{2}(\mathbf{M}^{(0)} + \mathbf{M}^{(1)})$ where $\mathbf{M}^{(0)}, \mathbf{M}^{(1)} \in \{0,1\}^{w \times w}$.
4. The derandomization task is equivalent to approximation of

$$\mathbf{M}^n = \mathop{\mathbb{E}}_{\sigma \sim \{0,1\}^n} \mathbf{M}^{(\sigma)}$$

   where $\mathbf{M}^{(\sigma)} = \mathbf{M}^{(\sigma_1)} \cdots \mathbf{M}^{(\sigma_n)}$.
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Redefining PRG and WPRG

An (n, w, ε) PRG $G : \{0,1\}^s \to \{0,1\}^n$ satisfies

$$\left\| \mathop{\mathbb{E}}_{\sigma \sim \{0,1\}^n} [\mathbf{M}^{(\sigma)}] - \mathop{\mathbb{E}}_{x \in \{0,1\}^s} [\mathbf{M}^{(G(x))}] \right\| \leq \varepsilon,$$

An (n, w, ε) WPRG $G = (I, \mu) : \{0,1\}^s \to \mathbb{R} \times \{0,1\}^n$ satisfies

$$\left\| \mathop{\mathbb{E}}_{\sigma \sim \{0,1\}^n} [\mathbf{M}^{(\sigma)}] - \mathop{\mathbb{E}}_{x \in \{0,1\}^s} [\mu(x) \cdot \mathbf{M}^{(I(x))}] \right\| \leq \varepsilon.$$

Encoding powers in Laplacians

Goal: given $\mathbf{M} \in \mathbb{R}^{w \times w}$, $\varepsilon > 0$, output $\widetilde{\mathbf{M}^n}$ s.t. $\left\| \widetilde{\mathbf{M}^n} - \mathbf{M}^n \right\| \leq \varepsilon$.

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Inverse of Laplacians

If $\mathbf{I} - \mathbf{A}$ is invertible, then

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \ldots + \mathbf{A}^n + \ldots$$

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Traced back to [Coo85], there is a simple reduction of "Laplacian inverse $\Longrightarrow$ Matrix powering":

$$\mathbf{P}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathbf{P}_4 \otimes \mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{M} & 0 & 0 & 0 \\ 0 & \mathbf{M} & 0 & 0 \\ 0 & 0 & \mathbf{M} & 0 \end{pmatrix}, \ (\mathbf{P}_4 \otimes \mathbf{M})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{M}^2 & 0 & 0 & 0 \\ 0 & \mathbf{M}^2 & 0 & 0 \end{pmatrix}$$

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so since $(\mathbf{A} \otimes \mathbf{B})^k = \mathbf{A}^k \otimes \mathbf{B}^k$ and $\mathbf{P}_{n+1}^{n+1} = 0$,

$$\begin{aligned} (\mathbf{I} - \mathbf{P}_{n+1} \otimes \mathbf{M})^{-1} &= \mathbf{I} + (\mathbf{P}_{n+1} \otimes \mathbf{M}) + (\mathbf{P}_{n+1} \otimes \mathbf{M})^2 + \ldots (\mathbf{P}_{n+1} \otimes \mathbf{M})^n + \ldots \\ &= \mathbf{I} + (\mathbf{P}_{n+1} \otimes \mathbf{M}) + \ldots + (\mathbf{P}_{n+1} \otimes \mathbf{M})^n \\ &= \mathbf{I} + (\mathbf{P}_{n+1} \otimes \mathbf{M}) + \ldots + (\mathbf{P}_{n+1}^n \otimes \mathbf{M}^n) \end{aligned}$$

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Encoding powers in Laplacians (cont.)

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As an example,

$$(\mathbf{I} - \mathbf{P}_4 \otimes \mathbf{M})^{-1} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{M} & \mathbf{I} & 0 & 0 \\ \mathbf{M}^2 & \mathbf{M} & \mathbf{I} & 0 \\ \mathbf{M}^3 & \mathbf{M}^2 & \mathbf{M} & \mathbf{I} \end{pmatrix}$$

Encoding powers in Laplacians (cont.)

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And using some notations:

1. Denote $\mathbf{L}\stackrel{\text{def}}{=}\mathbf{I}-\mathbf{P}_{n+1}\otimes\mathbf{M}$,
2. So we are interested in approximating $\mathbf{L}^{-1}$ (that contains $\mathbf{M}^n$).

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Lemma (Precondition Richardson)

Let $\mathbf{L} \in \mathbb{R}^{w \times w}$ *invertible matrix. Let* $\widetilde{\mathbf{L}^{-1}}$ *s.t.*

$$\left\|\widetilde{\mathbf{L}^{-1}} - \mathbf{L}^{-1}\right\| \leq \boldsymbol{\varepsilon_0}.$$

Define $\mathbf{R}_k \stackrel{\text{def}}{=} \sum_{i=0}^{k}(\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^i \cdot \widetilde{\mathbf{L}^{-1}}$. *Then,*

$$\left\|\mathbf{R}_k - \mathbf{L}^{-1}\right\| \leq \left\|\mathbf{L}^{-1}\right\| \cdot \|\mathbf{L}\|^{k+1} \cdot \boldsymbol{\varepsilon_0}^{k+1}.$$

Richardson Iterations

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The intuition is pretty simple, as for $k = \infty$,

$$\mathbf{R}_\infty = \left(\sum_{i=0}^{\infty} (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^i \right) \cdot \widetilde{\mathbf{L}^{-1}} = \left(\mathbf{I} - \left(\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L} \right) \right)^{-1} \cdot \widetilde{\mathbf{L}^{-1}} = \left(\widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L} \right)^{-1} \cdot \widetilde{\mathbf{L}^{-1}} = \mathbf{L}^{-1}$$

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Richardson Iterations (cont.)

And for arbitrary k, use geometric sum:

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Richardson Iterations (cont.)

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$$\mathbf{R}_k = \sum_{i=0}^{k} (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^i \cdot \widetilde{\mathbf{L}^{-1}} = \frac{\mathbf{I} - (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^{k+1}}{\mathbf{I} - (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})} \cdot \widetilde{\mathbf{L}^{-1}} = (\mathbf{I} - (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^{k+1}) \cdot \mathbf{L}^{-1}$$

Thus,

$$\begin{aligned} \left\| \mathbf{R}_k - \mathbf{L}^{-1} \right\| &= \left\| (\mathbf{I} - (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^{k+1}) \cdot \mathbf{L}^{-1} - \mathbf{L}^{-1} \right\| \\ &\leq \left\| (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^{k+1} \right\| \cdot \left\| \mathbf{L}^{-1} \right\| \\ &\leq \left\| \mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L} \right\|^{k+1} \cdot \left\| \mathbf{L}^{-1} \right\| \\ &\leq \left\| (\mathbf{L}^{-1} - \widetilde{\mathbf{L}^{-1}}) \cdot \mathbf{L} \right\|^{k+1} \cdot \left\| \mathbf{L}^{-1} \right\| \\ &\leq (\boldsymbol{\varepsilon_0} \cdot \|\mathbf{L}\|)^{k+1} \cdot \left\| \mathbf{L}^{-1} \right\| \end{aligned}$$

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Error reduction recipe

Richardson in one line

$$\left\|\widetilde{\mathbf{L}^{-1}} - \mathbf{L}^{-1}\right\| \leq \boldsymbol{\varepsilon_0} \quad \Longrightarrow \quad \left\|\mathbf{R}_k - \mathbf{L}^{-1}\right\| \leq \left\|\mathbf{L}^{-1}\right\| \cdot \|\mathbf{L}\|^{k+1} \cdot \boldsymbol{\varepsilon_0}^{k+1} \stackrel{\text{def}}{=} \varepsilon$$

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Richardson in one line

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Recall that given BP $\mathbf{M} \in \mathbb{R}^{w \times w}$, we wish to approximate $\mathbf{M}^n$. So...

Error reduction recipe

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Error reduction recipe

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Error reduction recipe

Richardson in one line

$$\left\|\widetilde{\mathbf{L}^{-1}} - \mathbf{L}^{-1}\right\| \le \boldsymbol{\varepsilon_0} \implies \left\|\mathbf{R}_k - \mathbf{L}^{-1}\right\| \le \left\|\mathbf{L}^{-1}\right\| \cdot \|\mathbf{L}\|^{k+1} \cdot \boldsymbol{\varepsilon_0}^{k+1} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}$$

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Error reduction recipe

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4. Output the bottom left block so $\|(\mathbf{R}_k)[n+1, 1] - \mathbf{M}^n\| \leq \boldsymbol{\varepsilon}$.

Error reduction recipe

Richardson in one line

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4. Output the bottom left block so $\|(\mathbf{R}_k)[n+1, 1] - \mathbf{M}^n\| \leq \varepsilon$.

Examples

Consider $k = 1$ and $n = 3$. Then,

$$\mathbf{L} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ -\mathbf{M} & \mathbf{I} & 0 & 0 \\ 0 & -\mathbf{M} & \mathbf{I} & 0 \\ 0 & 0 & -\mathbf{M} & \mathbf{I} \end{pmatrix}, \widetilde{\mathbf{L}^{-1}} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ \widetilde{\mathbf{M}} & \mathbf{I} & 0 & 0 \\ \widetilde{\mathbf{M}^2} & \widetilde{\mathbf{M}} & \mathbf{I} & 0 \\ \widetilde{\mathbf{M}^3} & \widetilde{\mathbf{M}^2} & \widetilde{\mathbf{M}} & \mathbf{I} \end{pmatrix}$$

and

$$\mathbf{R}_{k=1} = \widetilde{\mathbf{L}^{-1}} + (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^1 \cdot \widetilde{\mathbf{L}^{-1}}.$$

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We examine $\mathbf{R}_{k=1}$:

$$(\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{M} - \widetilde{\mathbf{M}} & 0 & 0 & 0 \\ \widetilde{\mathbf{M}}\mathbf{M} - \widetilde{\mathbf{M}^2} & \mathbf{M} - \widetilde{\mathbf{M}} & 0 & 0 \\ \widetilde{\mathbf{M}^2}\mathbf{M} - \widetilde{\mathbf{M}^3} & \widetilde{\mathbf{M}}\mathbf{M} - \widetilde{\mathbf{M}^2} & \mathbf{M} - \widetilde{\mathbf{M}} & 0 \end{pmatrix}$$

$$\mathbf{R}_{k=1} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{M} & \mathbf{I} & 0 & 0 \\ \widetilde{\mathbf{M}}\mathbf{M} + \mathbf{M}\widetilde{\mathbf{M}} - \widetilde{\mathbf{M}^2} & \mathbf{M} & \mathbf{I} & 0 \\ \widetilde{\mathbf{M}^2}\mathbf{M} - \widetilde{\mathbf{M}^2}\widetilde{\mathbf{M}} + \widetilde{\mathbf{M}}\mathbf{M}\widetilde{\mathbf{M}} - \widetilde{\mathbf{M}}\widetilde{\mathbf{M}^2} + \mathbf{M}\widetilde{\mathbf{M}^2} & \widetilde{\mathbf{M}}\mathbf{M} + \mathbf{M}\widetilde{\mathbf{M}} - \widetilde{\mathbf{M}^2} & \mathbf{M} & \mathbf{I} \end{pmatrix}$$

Examples (cont.)

Now take $k = 2$ and $n = 3$. Then,

$$\mathbf{L} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ -\mathbf{M} & \mathbf{I} & 0 & 0 \\ 0 & -\mathbf{M} & \mathbf{I} & 0 \\ 0 & 0 & -\mathbf{M} & \mathbf{I} \end{pmatrix}, \ \widetilde{\mathbf{L}^{-1}} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ \widetilde{\mathbf{M}} & \mathbf{I} & 0 & 0 \\ \widetilde{\mathbf{M}^2} & \widetilde{\mathbf{M}} & \mathbf{I} & 0 \\ \widetilde{\mathbf{M}^3} & \widetilde{\mathbf{M}^2} & \widetilde{\mathbf{M}} & \mathbf{I} \end{pmatrix}$$

and

$$\mathbf{R}_{k=2} = \widetilde{\mathbf{L}^{-1}} + (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^1 \cdot \widetilde{\mathbf{L}^{-1}} + (\mathbf{I} - \widetilde{\mathbf{L}^{-1}} \cdot \mathbf{L})^2 \cdot \widetilde{\mathbf{L}^{-1}}$$

Examples (cont.)

Now take $k = 2$ and $n = 3$. Then,

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So $\mathbf{R}_{k=2}$ looks like:

$$\mathbf{R}_{k=2} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{M} & \nwarrow & 0 & 0 \\ \mathbf{M}^2 - \mathbf{M}\widetilde{\mathbf{M}} + \widetilde{\mathbf{M}}^2 + \widetilde{\mathbf{M}}\mathbf{M} - \widetilde{\mathbf{M}^2} & \nwarrow & \nwarrow & 0 \\ \star & \nwarrow & \nwarrow & \nwarrow \end{pmatrix}$$

$$\star = \widetilde{\mathbf{M}}\mathbf{M}^2 - 2\widetilde{\mathbf{M}^2}\mathbf{M} + 3\widetilde{\mathbf{M}}^3 + \mathbf{M}\widetilde{\mathbf{M}}\mathbf{M} - \mathbf{M}\widetilde{\mathbf{M}}^2$$
$$+\mathbf{M}^2\widetilde{\mathbf{M}} + \widetilde{\mathbf{M}^2}\mathbf{M} - \widetilde{\mathbf{M}^2}\widetilde{\mathbf{M}} - \widetilde{\mathbf{M}}\widetilde{\mathbf{M}^2} + \mathbf{M}\widetilde{\mathbf{M}^2}$$

Deducing the improved accuracy of $\mathbf{M}^n$

Reminder

$\mathbf{M}$ is some BP, $\mathbf{L} \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{P}_{n+1} \otimes \mathbf{M}$, and

$$\mathbf{L}^{-1} = \begin{pmatrix} \mathbf{M}^0 & 0 & 0 \\ \vdots & \ddots & 0 \\ \mathbf{M}^n & \cdots & \mathbf{M}^0 \end{pmatrix}, \widetilde{\mathbf{L}^{-1}} = \begin{pmatrix} \widetilde{\mathbf{M}^0} & 0 & 0 \\ \vdots & \ddots & 0 \\ \widetilde{\mathbf{M}^n} & \cdots & \widetilde{\mathbf{M}^0} \end{pmatrix}$$

where $\left\| \widetilde{\mathbf{M}^i} - \mathbf{M}^i \right\| \leq \boldsymbol{\varepsilon_0}/n$.

Richardson

$$\left\| \widetilde{\mathbf{L}^{-1}} - \mathbf{L}^{-1} \right\| \leq \boldsymbol{\varepsilon_0} \quad \Longrightarrow \quad \left\| \mathbf{R}_k - \mathbf{L}^{-1} \right\| \leq \left\| \mathbf{L}^{-1} \right\| \cdot \left\| \mathbf{L} \right\|^{k+1} \cdot \boldsymbol{\varepsilon_0}^{k+1} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}.$$

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It's not hard to convince that

$$\|\mathbf{L}\| \leq 2 \quad , \quad \left\| \mathbf{L}^{-1} \right\| \leq n + 1 \leq 2n$$

Deducing the improved accuracy of $\mathbf{M}^n$

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$\mathbf{M}$ is some BP, $\mathbf{L} \overset{\text{def}}{=} \mathbf{I} - \mathbf{P}_{n+1} \otimes \mathbf{M}$, and

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thus,

$$\left\| (\mathbf{R}_k - \mathbf{L}^{-1})_{n+1,1} \right\| \leq \left\| \mathbf{R}_k - \mathbf{L}^{-1} \right\| \leq 2n \cdot (2 \cdot \boldsymbol{\varepsilon_0})^{k+1} \overset{\text{def}}{=} \boldsymbol{\varepsilon},$$

so we need to take $k \overset{\text{def}}{=} \frac{\log \boldsymbol{\varepsilon}^{-1}}{\log(4n^2/\boldsymbol{\varepsilon_0})}$.

Deducing the improved accuracy of $\mathbf{M}^n$

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where $\left\| \widetilde{\mathbf{M}^i} - \mathbf{M}^i \right\| \leq \boldsymbol{\varepsilon_0} / n$.

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Implied WPRG

Let $G : \{0,1\}^{s_0} \to \{0,1\}^n$ be some $\boldsymbol{\varepsilon_0}$ PRG, and denote $G_i(x) = G(x)_{0,\ldots,i-1}$.

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and for general k, it can be shown easily that

$$\mathbf{R}_k[n+1,1] = \sum_{r=1}^{n^k} \left(\pm \prod_{j_r=1}^{O(k)} \widetilde{\mathbf{M}^{i_{j_r}}} \right) = \sum_{r=1}^{n^k} \left(\pm \prod_{j_r=1}^{O(k)} \mathop{\mathbb{E}}_{x_{j_r} \in \{0,1\}^{s_0}} [\mathbf{M}^{(G_{i_{j_r}}(x_{j_r}))}] \right)$$

Implied WPRG

Let $G:\{0,1\}^{s_0}\to\{0,1\}^n$ be some ε_0 PRG, and denote $G_i(x)=G(x)_{0,\dots,i-1}$.
Recall,

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So how much does it cost to sample k **different** seeds from G?

Implied WPRG (cont.)

Let's say we take G as [Nis92] (actually, it's the best we can...),

Nisan [Nis92] PRG

For every n, w, ε there exists PRG against (n, w) BP with seed

$$O(\log n \cdot (\log n + \log w + \log \varepsilon^{-1}))$$

Recall that we need $\boldsymbol{\varepsilon_0} \stackrel{\text{def}}{=} 1/4n^2$, so

$$\begin{aligned} s_0 &= O(\log n \cdot (\log n + \log w + \log \boldsymbol{\varepsilon_0}^{-1})) \\ &= O(\log n \cdot (\log n + \log w)). \end{aligned}$$

Implied WPRG (cont.)

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Since we need $O(k)$ different seeds, where

$$O(k) = O\left(\frac{\log \boldsymbol{\varepsilon}^{-1}}{\log \boldsymbol{\varepsilon_0}^{-1}}\right) = O\left(\frac{\log \boldsymbol{\varepsilon}^{-1}}{\log n}\right)$$

the new seed length $\mathsf{s}_{\mathsf{new}}$ is

$$\begin{aligned} \mathsf{s}_{\mathsf{new}} &= O(k \cdot s_0) \\ &= O\left(\frac{\log \boldsymbol{\varepsilon}^{-1}}{\log n} \cdot \log n \cdot (\log n + \log w)\right) \\ &= \log \boldsymbol{\varepsilon}^{-1} \cdot (\log n + \log w) \end{aligned}$$

and sadly, we've got longer seed $\mathsf{s}_{\mathsf{new}} \gg s_0$, as $\boldsymbol{\varepsilon} \ll 1/n$...

Derandomizing the seeds selection

Recall that

$$\mathbf{R}_k[n+1,1] = \sum_{r=1}^{n^k} \left(\pm \prod_{j_r=1}^{O(k)} \mathop{\mathbb{E}}_{x_{j_r} \in \{0,1\}^{s_0}} [\mathbf{M}^{(G_{i_{j_r}}(x_{j_r}))}] \right)$$

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Reminder

Let $\mathbf{M}$ be an $(n, w, \Sigma = \{0,1\})$ BP with all identical layers. Our goal is to approximate $\mathbf{M}^n$ where,

$$\mathbf{M}^n = \left(\frac{1}{2} \left(\mathbf{M}^{(0)} + \mathbf{M}^{(1)} \right) \right)^n = \prod_{i=1}^{n} \mathop{\mathbb{E}}_{i \in \{0,1\}} \mathbf{M}^{(i)}$$

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Impagliazzo, Nisan, and Wigderson [INW94] PRG

There exists $(n', w', \Sigma, \varepsilon_{\mathrm{INW}})$ PRG with seed length

$$\mathsf{s}_{\mathsf{INW}} = O(\log n' \cdot (\log n' + \log w' + \log \varepsilon_{\mathrm{INW}}^{-1})) + \log |\Sigma|$$

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thus, the INW seed is:

$$\begin{aligned} s_{\text{INW}} &= O(\log k \cdot (\log k + \log w + \log \varepsilon_{\text{INW}}^{-1})) + s_0 \\ &= s_0 + O((\log \boldsymbol{\varepsilon}^{-1} + \log w) \cdot \log\log(1/\boldsymbol{\varepsilon})). \end{aligned}$$

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So we conclude an $\boldsymbol{\varepsilon}$ Weighted PRG against $(n, \boldsymbol{w})$ with seed length

$$\mathsf{s}_{\mathsf{new}} = \mathsf{s}_{\mathsf{INW}} = s_0 + \widetilde{O}(\log \boldsymbol{\varepsilon}^{-1} + \log w). \qquad ☺$$

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Subsequent work

Further improvement. Hoza [Hoz21] showed how to combine an idea from Armoni [Arm98] to remove the $\log\log$ factors for PRGs with *inherent gap* between their seed length and space complexity.
Since Nisan's PRG indeed "blessed" with that gap, he concluded a WPRG with seed length

$$O(\log n \cdot (\log n + \log w) + \log \varepsilon^{-1})$$

Error reduction from $\varepsilon_0 \gg 1/n$. For the restricted class of *permutation* branching programs, Pyne and Vadhan [PV21] established the analysis under another norm $\|\cdot\|$, and so concluded error reduction when $\varepsilon_0 = 1/\log n$.

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Future Directions

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2. Eliminate the restriction of the error reduction from $\boldsymbol{\varepsilon_0} \approx 1/n$
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Thanks!