Localization

Introduction to Algebraic-Geometric Codes. Fall 2019

April 30, 2019

Discussion

Let A be a domain. Let $0 \neq s \in A$ be an element we wish to "invert". We would like to enlarge A as little as possible so to include "an inverse" for s.

There are two ways of thinking about this:

1. *Working from "A upwards": construct the elements we need and add them to A.*
2. *Working from "Frac(A) downwards": picking only the elements we need from the field of fractions.*

We'll do both and establish that these are (strongly) equivalent.

Note that however will choose to invert s, we will "automatically" invert $s^2, s^3, \ldots$. More generally, if we wish to invert a pair of elements s, t, then we also invert $s \cdot t, s^2 \cdot t^5, \ldots$. This leads to the following definition.

Definition

Let A be a domain. A set $S \subset A$ is called multiplicative if

1. $0 \notin S$;
2. $1 \in S$; and
3. $\forall a, b \in S$, $ab \in S$.

Example

- The set of units $A^\times$ of A is multiplicative.
- $S = A \setminus \{0\}$ is multiplicative.
- If $P \in \mathrm{Spec}(A)$ then $S = A \setminus P$ is multiplicative.
- If $0 \neq a \in A$ then $S = \{a^i \mid i \in \mathbb{N}\}$ is multiplicative.

Construction ($S^{-1}A$)

Let A be a domain and $S \subset A$ a multiplicative subset of A. We construct the domain $S^{-1}A$ which we, informally, think of as the smallest ring obtained by adjoining to A the inverses of elements in S.

First, we define a relation on $A \times S$ as follows:

$$(a, s) \equiv (b, t) \iff at = bs.$$

Note that this is an equivalence relation on $A \times S$. We denote the equivalence class of (a, s) by $\frac{a}{s}$. The set of equivalence classes is denoted by $S^{-1}A$.

Construction ($S^{-1}A$ cont.)

We endow the set $S^{-1}A$ with a ring structure, as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{ts}$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{ts}.$$

The identity element of the ring is $\frac{1}{1}$.

One needs to verify, using that S is multiplicative, that $S^{-1}A$ is indeed a ring. It is called the ring of fractions of A with respect to S. In fact, $S^{-1}A$ is a domain.

Remark

When $S = A \setminus \{0\}$ *we get* $S^{-1}A = \mathrm{Frac}(A)$.

Remark

One can embed $A \hookrightarrow S^{-1}A$ *via the map* $j_S : A \hookrightarrow S^{-1}A$ *that sends* $a \mapsto \frac{a}{1}$. *This map, sometimes denoted simply by* j, *has the following properties:*

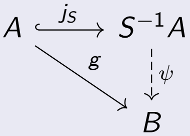
- j *is a ring monomorphism.*
- *For* $s \in S$, *the element* $j(s) = \frac{s}{1}$ *is invertible in* $S^{-1}A$ *with inverse* $\frac{1}{s}$. *Indeed,* $\frac{1}{s}, \frac{s}{1} \in S^{-1}A$ *and* $\frac{s}{1} \cdot \frac{1}{s} = \frac{s}{s} = \frac{1}{1}$ *since* $s \cdot 1 = s \cdot 1$ *(in* A*).*

Proposition (Universal property of rings of fractions)

Let A be a domain and $S \subset A$ a multiplicative subset. Let $g : A \to B$ be a ring homomorphism s.t.

$$\forall s \in S \quad g(s) \in B^\times$$

Then, there exists a unique ring homomorphism $\psi\colon S^{-1}A \to B$ s.t. $g = \psi \circ j_S$.



Proof.

We define

$$\psi\left(\frac{a}{s}\right) = g(a) \cdot g(s)^{-1}.$$

ψ is well-defined. Indeed, if $\frac{a}{s} = \frac{a'}{s'}$ (in $S^{-1}A$) then

$$\implies as' = a's \quad (\text{in } A).$$

$$\implies g(as') = g(a's) \quad (\text{in } B)$$

$$\implies g(a)g(s') = g(a')g(s) \quad (\text{in } B)$$

$$\implies g(a)g(s)^{-1} = g(a')g(s')^{-1}$$

$$\implies \psi\left(\frac{a}{s}\right) = \psi\left(\frac{a'}{s'}\right).$$

□

Proof (cont.)

As for uniqueness, let $\psi : S^{-1}A \to B$ be a ring homomorphism such that $g = \psi \circ j_S$. Then, for every $a \in A$

$$\psi\left(\frac{a}{1}\right) = (\psi \circ j_S)(a) = g(a).$$

For every $s \in S$,

$$1 = \psi(1) = \psi\left(\frac{s}{1} \cdot \frac{1}{s}\right) = \psi\left(\frac{s}{1}\right) \cdot \psi\left(\frac{1}{s}\right) = g(s) \cdot \psi\left(\frac{1}{s}\right).$$

Thus, $\forall s \in S \ \psi(\frac{1}{s}) = g(s)^{-1}$. Therefore,

$$\psi\left(\frac{a}{s}\right) = \psi\left(\frac{a}{1}\right) \cdot \psi\left(\frac{1}{s}\right) = g(a) \cdot g(s)^{-1}.$$

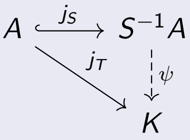
□

Corollary

Let A be a domain with $K = \mathrm{Frac}(A)$. Let $S \subset A$ be a multiplicative subset. Then, the ring $S^{-1}A$ can be identified in a unique manner with the subring $A[\{\frac{1}{s} \mid s \in S\}]$ of K.

Proof.

Recall that $\mathrm{Frac}(A) = T^{-1}A$ where $T = A \setminus \{0\}$. Consider



Since $j_T(S) \subseteq K^\times \quad \exists! \psi : S^{-1}A \to K$ s.t. $j_T = \psi \circ j_S$. Note that ψ is injective and so $S^{-1}A$ can be identified with $\psi(S^{-1}A)$ in a unique manner. □

Proof (cont.)

If we identify A with its image under $j_T(A)$ then

$$\psi\left(\frac{a}{s}\right) = j_T(a) \cdot j_T(s)^{-1} = \frac{a}{s}.$$

Therefore, $\psi(S^{-1}A) = A[\{\frac{1}{s} \mid s \in S\}]$. $\square$

Discussion

There seem to be two natural ways to "construct" an ideal in $S^{-1}A$ from an ideal I of A:

1. *consider the ideal generated by the image $j_S(I)$, namely, $j_S(I)(S^{-1}A)$; or*
2. *consider the ideal $\{\frac{i}{s} \mid i \in I, s \in S\}$.*

Claim

The two definitions coincide.

Definition

We denote the ideal generated by the image $j_S(I)$ by $S^{-1}I$.

Proof.

Let $i \in I, s \in S$. Then, $\frac{i}{1} \in j_S(I)$ and $\frac{1}{s} \in S^{-1}A$. Thus,

$$\frac{i}{s} = \frac{i}{1} \cdot \frac{1}{s} \in j_S(I)(S^{-1}A) = S^{-1}I.$$

As for the other direction, an element in $S^{-1}I$ is of the form

$$\sum_{j=1}^{n} \frac{i_j}{1} \cdot \frac{a_j}{s_j} = \frac{1}{s} \cdot \sum_{j=1}^{n} i_j a_j t_j,$$

where $s = s_1 \cdots s_n$ and t_j is the product of all $s_1, \ldots, s_n$ excluding s_j. The proof follows since $i_j a_j t_j \in I$. □

Claim

$S^{-1}I = S^{-1}A$ if and only if $I \cap S \neq \emptyset$.

Proof.

$$S^{-1}I = S^{-1}A \iff \frac{1}{1} \in S^{-1}I$$

By the previous claim, the above holds $\iff \exists i \in I, s \in S$ s.t. $\frac{i}{s} = \frac{1}{1}$. Namely, if and only if $i = s$ (in A). This is equivalent to $I \cap S \neq \emptyset$. □

Claim

Let A be a domain and $S \subset A$ a multiplicative subset. Let $P \in \operatorname{Spec}(A)$ be such that $P \cap S = \emptyset$. Then, $S^{-1}P \in \operatorname{Spec}(S^{-1}A)$.

Proof.

Let $x_1 = \frac{a_1}{s_1}, x_2 = \frac{a_2}{s_2} \in S^{-1}A$ s.t. $x_1 x_2 = \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} \in S^{-1}P$. By a previous claim

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{p}{s}$$

for some $p \in P, s \in S$. Thus, in A we have that

$$a_1 a_2 s = s_1 s_2 p \in P$$

Now, $s \in S \implies s \notin P \implies a_1 a_2 \in P \implies a_1 \in P$ (w.l.o.g) $\implies x_1 = \frac{a_1}{s_1} \in S^{-1}P$. □

Definition

Let A be a domain and $S \subset A$ a multiplicative subset. Define the map

$$\mathcal{J} \colon \{P \in \mathrm{Spec}(A) \mid P \cap S = \emptyset\} \to \mathrm{Spec}(S^{-1}A)$$
$$P \mapsto S^{-1}P$$

Remark

The previous claim tells us that $\mathcal{J}$ is well defined.

Claim

Let A, S as above. The map $\mathcal{J}$ is a bijection.

Proof.

Surjective. Take $Q \in \mathrm{Spec}(S^{-1}A)$. Define

$$P = \left\{ a \in A \mid \frac{a}{1} \in Q \right\}.$$

$P \in \mathrm{Spec}(A)$ since

$$p_1, p_2 \in P \implies \frac{p_1}{1}, \frac{p_2}{1} \in Q \implies \frac{p_1}{1} + \frac{p_2}{1} = \frac{p_1 + p_2}{1} \in Q$$

and so $p_1 + p_2 \in P$. Further,

$$p \in P, a \in A \implies \frac{ap}{1} = \frac{a}{1} \cdot \frac{p}{1} \in Q$$

and so $ap \in P$. □

Proof.

Surjective (cont.) We are left to show that $P \cap S = \emptyset$. Otherwise take $s \in S \cap P$.

$$s \in P \implies \frac{s}{1} \in Q.$$

$$s \in S \implies \frac{1}{s} \in S^{-1}A.$$

Hence,

$$\frac{1}{1} = \frac{s}{1} \cdot \frac{1}{s} \in Q$$

in contradiction to Q being prime. □

Proof.

Injective (cont.) Take $P_1, P_2 \in \mathrm{Spec}(A)$ with $P_1 \cap S = P_2 \cap S = \emptyset$ and assume that $S^{-1}P_1 \subseteq S^{-1}P_2$. Then, $\forall p_1 \in P_1, s_1 \in S_1 \; \exists p_2 \in P_2, s_2 \in S_2$ s.t. $\frac{p_1}{s_1} = \frac{p_2}{s_2}$. So, $p_1 s_2 = p_2 s_1$ in A. Since $p_2 s_1 \in P_2$ and $s_2 \notin P_2$ we have $p_1 \in P_2$. So, $P_1 \subseteq P_2$. Similarly, $S^{-1}P_2 \subseteq S^{-1}P_1 \implies P_2 \subseteq P_1$. Thus, $S^{-1}P_1 = S^{-1}P_2 \implies P_1 = P_2$. □

Remark

Observe that the map $\mathcal{J}$ is inclusion-preserving.

Definition

Let A be a domain and $P \in \text{Spec}(A)$. Recall that $S = A \setminus P$ is multiplicative. We denote $S^{-1}A$ by A_P and call this domain the localization of A at P.

Claim

Let A be a domain and $P \in \text{Spec}(A)$. Then, $\text{Spec}(A_P)$ is in inclusion-preserving bijection with $\{Q \in \text{Spec}(A) \mid Q \subseteq P\}$.

Proof.

Follows immediately by the previous claim and since $S = A \setminus P$. □

Definition

A ring A is called a local ring if it has a unique maximal ideal.

Claim

Let A be a domain and $P \in \mathrm{Spec}(A)$. Then, A_P is a local ring with maximal ideal PA_P.

Proof.

$PA_P \in \mathrm{Spec}(A_P)$ and all ideals of A_P are contained in PA_P. $\square$

Corollary

Let A be a domain and $P \in \operatorname{Spec}(A)$. Then,

$$\operatorname{ht}(P) = \dim(A_P)$$

and

$$\dim(A) = \sup(\dim(A_P) \mid P \in \operatorname{Spec}(A)).$$

Corollary

Let A be a domain of dimension 1. Let $P \in \operatorname{Spec}(A)$. Then, $\dim(A_P) = 1$. More generally, if S is multiplicative such that $A \setminus S$ contains a prime ideal then $\dim(S^{-1}A) = 1$.

Discussion

Let $f \in \bar{K}[x, y]$ *irreducible. Take* $\alpha \in \bar{K}(Z_f)$ *and consider* $(a, b) \in Z_f(\bar{K})$. *Recall that*

$$\alpha \text{ is defined at } (a, b) \iff \exists g, h \in C_f \text{ s.t. } \alpha = \frac{g}{h} \text{ and } h(a, b) \neq 0$$

Let $P = \langle x - a, y - b \rangle \in \mathrm{Max}(C_f)$ *be the ideal that corresponds to* (a, b) *via Hilbert's Nullstellensatz. Recall that*

$$h \in P \iff h(a, b) = 0.$$

Thus,

$$\alpha \text{ is defined at } (a, b) \iff \exists g, h \in C_f \text{ s.t. } \alpha = \frac{g}{h} \text{ and } h \notin P \neq 0$$

Hence, $(C_f)_M$ *is uniquely identified with the subring of* $\bar{K}(Z_f)$ *consisting in all elements of* $\bar{K}(Z_f)$ *that are defined at* (a, b).

Generally, localization makes a ring simpler. You are asked to prove the following claim, which is an instantiation of this phenomena, in the homework assignment.

Claim

Let A be a domain and $S \subset A$ a multiplicative subset. Then,

- *A noetherian $\implies$ $S^{-1}A$ is noetherian.*
- *A integrally closed $\implies$ $S^{-1}A$ is integrally closed.*

Corollary

Let A be a Dedekind domain. Then, A_P is a Dedekind domain for every $P \in \mathrm{Spec}(A)$.

Claim

Let A be integrally closed domain contained in a field L. Let $S \subset A$ multiplicative. Let B denote the integral closure of A in L. Then, $S^{-1}B$ is the integral closure of $S^{-1}A$ in L.

Proof.

As B is integrally closed, $S^{-1}B$ is integrally closed. So, it suffices to show that $S^{-1}B$ is integral over $S^{-1}A$. An element of $S^{-1}B$ is of the form s/b for $s \in S, b \in B$. Since B is integral over A,

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$$

for some $a_0, \ldots, a_{n-1} \in A$. So,

$$\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s} \cdot \left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_0}{s^n} = 0$$

in $S^{-1}A$, proving that b/s is integral over $S^{-1}A$. □