Recitation 2: One and Two Qubits

2.1 A bit more general measurements

In the lecture you saw that a measurement can be defined by choosing an orthonormal basis for the space of states of a qudit (you saw it for a qubit, but it's true in general). There is a slight generalisation for that by choosing a decomposition of the space into orthogonal direct sum $V = S_1 \oplus S_2 \oplus \cdots \oplus S_m$. When performing the measurement we get two outcomes: the answer $i \in [m]$ to which subspace S_i the state collapsed (sometimes referred to as "the classical outcome") and the state itself changes by projecting the original state into S_i and renormalising it. This i outcome occurs with probability $\left\|\Pi_i \left|\psi\right\rangle\right\|^2$ where Π_i is the orthogonal projection onto S_i.

Using this kind of measurement, give a probabilistic way to create a uniform superposition over all the even computational basis elements $\frac{\sqrt{2}}{\sqrt{d}} \sum_{\substack{i \in \{0,\ldots,d-1\} \\ i \equiv 0 \mod 2}} |i\rangle$ (you can assume d is even), assuming we can create a uniform superposition over all the computational basis elements $\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$. What is the probability of your protocol to succeed?

Answer. If we have at hand $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$, then we can define the measurement that splits the vector space into odd and even computational basis elements $\mathbb{C}^d = span\{|i\rangle : i \equiv 0 \mod 2\} \oplus span\{|i\rangle : i \equiv 1 \mod 2\} =: S_0 \oplus S_1$. If the classical answer is 0 we succeeded. This happens with probability $\frac{1}{2}$. △

In the lecture we will see measurements of one qubit out of two. These can understood in terms of the more general measurements as a measurement of the whole space, that respects the qubits structure and decomposes the space according to one qubit, making a measurement that probes only the part of the one being measured: $\mathbb{C}^{2^2} = span\{|00\rangle, |01\rangle\} \oplus span\{|10\rangle, |11\rangle\}$.

2.2 Global VS Relative phases

In the exercise you were given two qudit states that differ by a global phase $|u\rangle = e^{\theta i} |v\rangle$ for $\theta \in \mathbb{R}$, and were told to show any measurement would give the same distribution on both. Let's use the more general measurement we just learned for that.

Answer. Let $V = S_1 \otimes \cdots \otimes S_m$ be a measurement for the qudit, and $\{\Pi_i\}_{i=1}^m$ the corresponding orthogonal projections. The distribution induced by performing the measurement on a state $|u\rangle$ is $(\|\Pi_1 |u\rangle\|^2, \cdots, \|\Pi_m |u\rangle\|^2)$. Using the relation between $|u\rangle$ and $|v\rangle$ we get $\|\Pi_i |u\rangle\|^2 = \left\|\Pi_i e^{\theta i} |v\rangle\right\|^2 = \left\|e^{\theta i}\Pi_i |v\rangle\right\|^2 = \left|e^{\theta i}\right|^2 \cdot \|\Pi_i |v\rangle\|^2 = \|\Pi_i |v\rangle\|^2$ so the induced distributions agree on $|u\rangle$ and $|v\rangle$. $\triangle$

You were also challenged to decide whether this means that anything we can do with $|u\rangle + e^{\theta_1 i} |v\rangle$ will have the same outcomes as with $|u\rangle + e^{\theta_2 i} |v\rangle$ for orthonormal $|u\rangle, |v\rangle$ and $\theta_1, \theta_2 \in \mathbb{R}$ are the relative phases.

Answer. We already saw this is not true: $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ whereas $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, and $|+\rangle, |-\rangle$ are orthonormal (here $\theta_1 = 0$ and $\theta_2 = \pi$) while a computational-basis measurement will give with certainty two different answers. $\triangle$

2.3 Distinguishing Two Qubit States

In the lecture we saw the Elizur-Vaidman bomb experiment. We had a qubit get into a black box that either did nothing or measured it in the computational basis, and ignited the bomb if the outcome was $|1\rangle$. We thought of inputting a qubit in a state close to $|0\rangle$ but not quite $|0\rangle$: $|\theta\rangle := \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ for some small angle θ. If we get to see the qubit out of the box, it is either $|\theta\rangle$ or $|0\rangle$. Could we just use this information and distinguish between these cases, without sending the qubit back to the box?

Another motivation is the amount of information we can load on one qubit: for any $\theta \in [0, \frac{\pi}{2}]$ we have a unique qubit state. Doesn't that mean we can store an infinite amount of information on just one qubit? In order to make the qubit state informational we have to at least be able to distinguish any two distinct qubit states $\theta \neq \theta'$ with high probability (for example two classical distributions on $\{0, 1\}$ can be distinct while we won't see their difference in any meaningful way). We would like to understand the limitations on this task.

We would like to distinguish between two specific qubit states $|u\rangle, |v\rangle$ in the sense that given $|\psi\rangle \in \{|u\rangle, |v\rangle\}$ we are tasked to pronounce which one we were given[1]. Let's understand how well we can perform this task. For convenience, assume $|u\rangle, |v\rangle$ have real amplitudes.

We ended here. Answers are suppressed below for now.

(a) Prove the only important factor here is the angle between the two vectors, $\langle u|v\rangle$.

[1] This is taken from a video lecture by Ryan O'Donnell

(b) Two-sided error: using the previous subquestion, we may decide to orient the vectors $|u\rangle, |v\rangle$ in the real plane, where the the bisector (the mid-angle) is at $|+\rangle$, where $|u\rangle$ is closer to $|0\rangle$ and $|v\rangle$ to $|1\rangle$ (we cannot change the order of $|u\rangle$ and $|v\rangle$, but clearly we can run the whole protocol with $|u\rangle$ being the one closer to $|1\rangle$, and the analysis will be similar). Then we can propose to measure in the computational basis and guess that if we got the outcome $|0\rangle$ we started with $|u\rangle$, and if we got the outcome $|1\rangle$ we started with $|v\rangle$. What kind of errors are possible in our protocol? Calculate the probability of an error in this protocol.

(c) One-sided error: propose a different protocol, where the error is asymmetric. We want to be infallible, meaning having no probability to err, if we get $|u\rangle$, and for that we allow ourselves to make mistakes if we are given $|v\rangle$, but hopefully with as little probability as possible. What is the error probability you can achieve, in case we get $|v\rangle$?

(d) "Zero-sided error": of course the previous subquestion could easily be made into a protocol that can err only if we are given $|v\rangle$. Use these two one-sided error protocols, one for each side, and construct a protocol that is allowed to guess also a don't-know answer, i.e. one of the three $\{|u\rangle, |v\rangle, \text{"don't know"}\}$, but when it guesses a vector it must be correct. Clearly we would have liked the probability of "don't know" to be minimal. What is the probability your procedure achieves?

Note that our "zero-sided" error is a bit weird, as at $\frac{\pi}{2}$ angle, when we can distinguish perfectly, we get probability $\frac{1}{2}$ for an error. This can be avoided later on, when we can use another qubit for the measurement.

To conclude, let's see all the error-probability curves together so we can compare them:

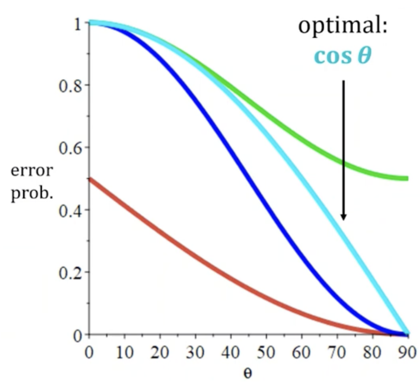

Figure 2.1: All the error probability curves together, by θ, the angle between $|u\rangle, |v\rangle$: two-sided error, one-sided, "zero-sided", and optimal "zero-sided"